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Dynamics of over-constrained rigid and flexible multibody systems

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Abstract — *the present paper deals with the problems of dynamic simulation of over-constrained multibody systems applied in different motion and power transferring mechanical devices, self-locking mechanisms, etc. The elasticity of the links is taken into account. Generalized Newton-Euler dynamic equations are applied for the case of finite element discretization of flexible links. An approach of decomposition of the mechanisms in the singular configurations is proposed. Relative and absolute nodal coordinates are used. The method substitutes the kinematic constraints by elastic forces. An example of dynamic analysis of over-constrained mechanisms is presented.*

Keywords: dynamics simulation, over-constrained mechanisms, flexibility.

I. Introduction

The theory of the constrained dynamics is very well developed in the mechanics and widely applied in multibody system motion simulation [1 – 5]. The constraints imposed on the systems have different nature and type. Most often in the theory of the mechanisms kinematic and force constraints are regarded. Joints connecting rigid bodies impose kinematic constraints, while if one regards the elastic nature of the contact between the bodies the same joints could be regarded as force constraints. Closed chains in the mechanical scheme impose kinematic constraints that additionally diminish the system degree of freedom (dof). Furthermore, the notation "common constraints" is used for systems that have common restrictions imposed on the motion of the links. Such systems are: the plane mechanisms (restrictions for motion in the plane); spherical mechanisms (the joint axes are crossing in a common point); the mechanism of Bennett [6], etc.

The dynamics of constraint system is presented by Differential Algebraic Equations (DAE). Surveys of the existing techniques for solving DAE may be found in [4, 7]. The classical method to deal with DAE is to express the constraint condition at acceleration level. This leads to replacement of the original system by a system of Ordinary Differential Equations ODE. Maintaining the acceleration constraints one does not satisfy the position and velocity constraints. Baumgarte's stabilization [8] term is introduced to ensure exponential convergence of the constraint error to zero. The problem with this method is in selection of high gains to keep small constraint errors. A similar approach based on penalty functions is presented in [4, 9]. Implementation of both methods [4, 8, 9] results in inclusion additional terms in the right side of the dynamic equations that could be treated as reaction forces in the joints cut but cannot be compared to the elastic forces in links.

Another group of researchers [10, 11, 12] proposed projection techniques to maintain the constraint conditions without modification of the equations of motion.

Other methods are based on coordinate partitioning [13]. At every step the set of the coordinates is partitioned of dependent and independent coordinates. However, a fixed set of independent coordinates may lead to dependent matrix of the derivatives of the constraints [4, 11].

Manipulation of the dynamic equations in the singular configurations is a challenging realm of the investigations. The Augmented Lagrangian formulation proposed in [14, 15] can handle redundant constraints in singular configurations. In [16] an approach for kinematic analysis of mechanisms and their singular configurations using the Moore–Penrose pseudo-inverse matrix is applied. Eich-Soellner and Fuhrer [17] solved the problem of constraint stabilization using optimization algorithms and the pseudo-inverse matrix so derived. In [18] a projection method is applied for simulation of constrained multibody systems. Mechanisms in the vicinity of singular configurations are regarded. Friction is taken into account. In [19] a pseudo-inverse matrix is proposed for effective solution of DAE and its application in singular configurations. However, special singular configurations exist for which there is no general solution and special methods are to be developed taking into account the elasticity of the links.

In the paper an approach for dynamic analysis of multibody systems that provides general solution in singular configurations and self-locking position is proposed. Elasticity of the links is taken into account. Generalized Newton-Euler dynamic equations are applied for the case of finite element discretization of flexible links. Relative and absolute nodal coordinate formulation is used. The closed kinematic chains are decomposed in open chains substituting the kinematic constraints by elastic forces due to elasticity of the links. Several examples of closed chain mechanisms in singular configuration and self-locking position are discussed.

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II. Topology and kinematics of over-constrained mechanisms

Over-constrained mechanisms are systems which dof are less than degrees of mobility. For example, if one analyze a plain mechanism considering the regulations for the spatial mechanism you will obtain less dof (even negative), while it is quite applicable. So plane mechanisms could be also considered over-constrained, although in practice no one is thinking so. Spherical mechanisms are also over-constrained and it could be easily observed if the precision of the links and orientation of joint axes are not fulfilled within the prescribed tolerances. In Fig. 1 some of most famous over-constrained mechanisms are shown.

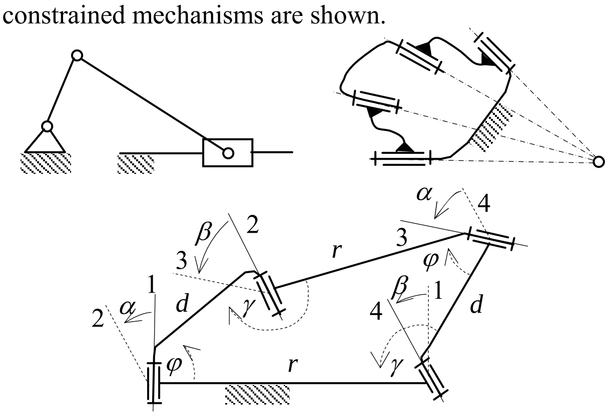

Fig. 1. Plane, spherical and Bennett's linkages regarded as over-constrained mechanisms

On the other hand, many closed chain mechanisms are movable, while the topology analysis shows zero or negative dof. This is because of the kinematic parameters for which the constraints equations are dependent. In Fig. 2 the simplest examples of such plane mechanisms are shown.

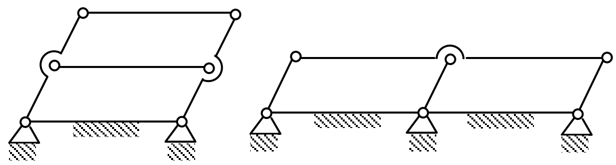

Fig. 2. Plane mechanisms with dependent constraints

In some mechanisms the proportion between the shape and size of the links is the reason for the increase of mechanism dof in specific positions, called singular configurations. Examples of mechanisms with closed chains in singular configuration for some basic groups from the classification of Assur are presented in Fig. 3.

The different nature and behavior of the over-constrained mechanism is the reason for the development of specific approaches for the kinematic and dynamic analysis and simulation for almost every single case. Even for a single mechanism it could happen that different approaches are to be applied during its motion. That

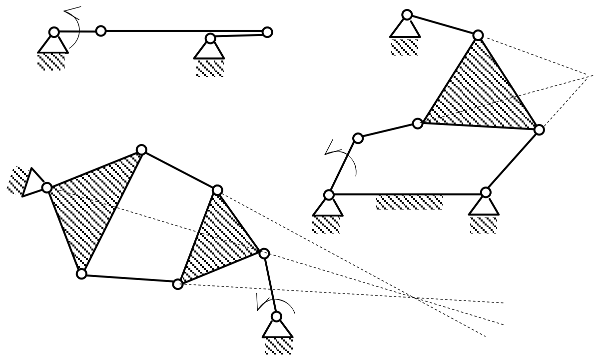

Fig. 3. Singular configurations for some basic groups of the Assur's classification

significantly slows down the effectiveness of the computations. But only for few cases such geometrical considerations could be regarded. For complex plane and even for simple space mechanisms, simple regulations cannot be discovered. For the case of singular configurations it is well know that the matrix of the derivatives of the constraint equation system (Jacobean matrix) is singular. This analysis is an onerous task, causes additional branches of the algorithms and cannot be implemented effectively in the vicinity of singularities. For numerical simulation of flexible system discretization of the continuum should be implemented and mass and stiffness properties of the flexible bodies are to be reduced to a finite number of points called nodes. The node of a flexible element is a free object that, in the three dimensional space, has six degrees of freedom [20]. The node motion is restricted by elastic forces acting between the neighbor nodes. In Figure 4, a flexible element with many nodes is shown, where for the node with index *i* the coordinates (using translations and Euler angles) of the node coordinate system relative to the

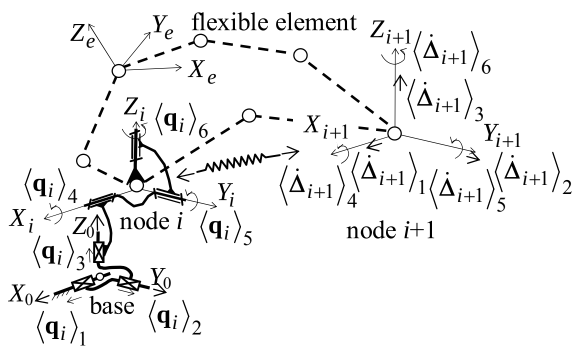

Fig. 4. Flexible finite element node coordinate systems

absolute reference frame are pointed out, while for the second node $(i+1)$ the linear and angular velocities of the node are shown. The coordinates of node *i* are stored in a 6×1 matrix $\mathbf{q}_i = \left[\langle\mathbf{q}_i\rangle_1 \quad \langle\mathbf{q}_i\rangle_2 \quad \cdots \quad \langle\mathbf{q}_i\rangle_6\right]^{\backslash}$, where the notations $\langle\mathbf{q}_i\rangle_m, m = 1,2,...,6$ are the elements of the matrix. "\" (backslash) denotes matrix transpose. The small finite translations and rotations of node i are compiled in a similar matrix Δ_i with elements $\langle \Delta_i \rangle_m, m = 1,2,...,6$. The kinematic characteristics of motions of the nodes are mutually independent. The nodes of flexible elements have six degrees of freedom either with respect to the element coordinate system ($X_e Y_e Z_e$) or to the absolute ($X_0 Y_0 Z_0$) one and their motions could be presented by virtual spatial joint with six dof as described in details in [20]. But it should be made clear difference between the coordinates $\mathbf{q}_i$ and the small possible motions Δ_i. The definition of the nodes as coordinate systems allows the flexible particles of the multibody systems, similarly to the systems of rigid bodies, to be decomposed to systems of moving coordinate systems connected by joints.

All (of number a) coordinates of the system are placed in matrix ${}^{a}\mathbf{Q}$. The left superscripts denote matrix dimension, i.e.: ${}^{i}\mathbf{A}$, ${}^{i,j}\mathbf{A}$, ${}^{i,j,k}\mathbf{A}$ are $i \times 1$, $i \times j$ and $i \times j \times k$ matrix–vector, plane and cubic matrices that, if once defined, could be missed. The coordinates $\mathbf{Q}$ are subject to constraints that define the function of $\mathbf{Q}$ with respect to generalized (of number g) coordinates ${}^{g}\mathbf{q}$.

The system is subject to d constraints ($d = a - g$), i.e.:

$$ {}^{d}\mathbf{\Phi} = \mathbf{\Phi} = \mathbf{\Phi}(\mathbf{Q}) = {}^{d}\mathbf{0} \quad (1) $$

where ${}^{d}\mathbf{0}$ is $d \times 1$ zero matrix. The time derivatives are:

$$ \dot{\mathbf{\Phi}} = \frac{\partial \mathbf{\Phi}}{\partial \mathbf{Q}} \cdot \dot{\mathbf{Q}} = {}_{\partial \mathbf{Q}}\mathbf{\Phi} \cdot \dot{\mathbf{Q}} = {}^{d}\mathbf{0} \quad (2) $$

$$ \ddot{\mathbf{\Phi}} = {}_{\partial \mathbf{Q}}\mathbf{\Phi} \cdot \ddot{\mathbf{Q}} + {}_{\partial^2 \mathbf{Q}}\mathbf{\Phi} \underset{1\backslash 3}{\otimes} \dot{\mathbf{Q}} \cdot \dot{\mathbf{Q}} = {}^{d}\mathbf{0} \quad (3) $$

where ${}_{\partial \mathbf{Q}}\mathbf{\Phi} = {}^{d,a}_{\partial \mathbf{Q}}\mathbf{\Phi}$ and ${}_{\partial^2 \mathbf{Q}}\mathbf{\Phi} = {}^{d,a,a}_{\partial^2 \mathbf{Q}}\mathbf{\Phi}$ are matrices of the first and second order partial derivatives (the left subscripts denote the differentiating variables). Notation "$\underset{1\backslash 3}{\otimes}$" presents matrix multiplication of three dimensional (space) matrix. Eq. 1 defines two sets of dependent ${}^{d}\underline{\mathbf{Q}}$ and independent ${}^{g}\mathbf{q}$ coordinates, i.e., $\mathbf{Q} = \left[\underline{\mathbf{Q}}^{\backslash} \quad \mathbf{q}^{\backslash} \right]^{\backslash}$.

The velocity equation, Eq. 2, could be transformed to [4]

$$ \dot{\mathbf{Q}} = \mathbf{R} \cdot \dot{\mathbf{q}} \quad (4) $$

and the time derivatives of Eq. 4 are as follows [20]:

$$ \ddot{\mathbf{Q}} = \mathbf{R} \cdot \ddot{\mathbf{q}} + {}_{\partial \mathbf{q}}\mathbf{R} \underset{1\backslash 3}{\otimes} \dot{\mathbf{q}} \cdot \dot{\mathbf{q}} \quad (5) $$

The matrices $\mathbf{R}$ and ${}_{\partial \mathbf{q}}\mathbf{R}$ are computed from the partial derivatives ${}_{\partial \mathbf{Q}}\mathbf{\Phi}$ and ${}_{\partial^2 \mathbf{Q}}\mathbf{\Phi}$ [20]. The equality constraints represent the connectivity of the links in closed chains. The equation constraints of closed chains contain dependent coordinates and, most often, they are the reason for the singularity of the Jacobean matrix. Equation constraints of open branches could be directly transformed with respect to the independent coordinates. This approach is widely applied in constraint dynamics [1 – 4] and consists in virtual disconnections of joints that pertain to the closed chain. The entire constraint equations system is then compiled inserting additional kinematic constraints that present the connectivity.

The principle proposed in the paper consists in transformation of a closed chain into open branch cutting not the joints but the flexible links. Using the finite element approach allows the kinematic constraints to be substituted by force constraints, i.e. by elastic forces in the nodes. Illustration of this approach applied for the mechanisms of Fig. 3 is presented in Fig. 5.

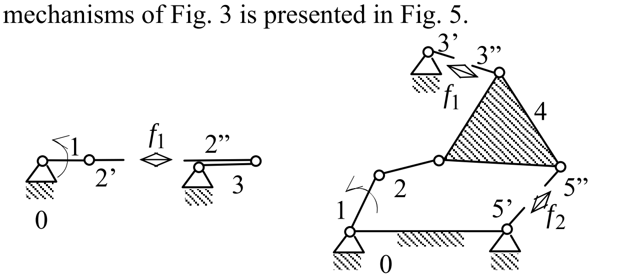

Fig. 5. Transformation of a closed chain into open branches substituting flexile links by elastic forces

Such transformation of a three contour six-link mechanism in Fig. 5 results in an open chain with four branches and eight links. The elastic forces f_1 and f_2 in Fig. 5 depend on the shape, size and stiffness of the links cut. The first step is the transformation of the closed chain into open branches representing the connectivity between the coordinate systems of the rigid bodies and the coordinate systems of the nodes of the flexible elements. For example, for the four-bar mechanism in Fig. 3 and its transformation (Fig. 5) the connectivity of the coordinate systems is shown in Fig. 6. The coordinate systems with

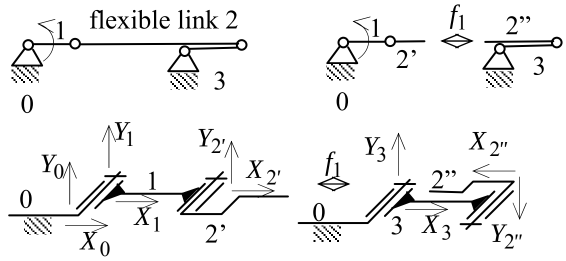

Fig. 6. Connectivity of the coordinate systems of rigid links and nodes

indices 2' and 2" correspond to the nodes of the flexible beam – link 2. The kinematic analysis of these two branches is a trivial task for every computer code generation program. For the next stage, the dynamic analysis, the main initial preparations will consist in mass distribution of the flexible elements to the node coordinate system (estimation of the mass matrix), as well as, the elastic forces (the stiffness matrix).

III. Dynamics of rigid and flexible mechanisms

Vector translation $\mathbf{s}_{C_i}$ of an object (point or node C_i of a body or a node of flexible element), and vector of the small rotations θ_i compile the matrix of the finite displacements $\Delta_i = \left[\mathbf{s}_{C_i}^{\backslash} \quad \theta_i^{\backslash}\right]^{\backslash}$. The coordinate transformation matrix, ${}^{6,6}\tau\Delta_i$, for vector Δ_i is:

$$\tau\Delta_i(\mathbf{q}) = \begin{bmatrix} \tau_i & {}^{3,3}\mathbf{0} \\ {}^{3,3}\mathbf{0} & \tau_i \end{bmatrix} \tag{6}$$

where ${}^{3,3}\tau_i$ is the matrix transformation of the coordinate system i. Similarly to Δ_i and Eqs. 4, 5, the quasi velocities and accelerations $\dot{\Delta}_i$, $\ddot{\Delta}_i$ are expressed with respect to $\mathbf{q}$, as well as, $\dot{\mathbf{q}}$ and $\ddot{\mathbf{q}}$ [20], i.e.:

$$\dot{\Delta}_i(\mathbf{q},\dot{\mathbf{q}}) = \begin{bmatrix} \mathbf{v}_{C_i} \\ \omega_i \end{bmatrix} = \mathbf{R}\Delta_i(\mathbf{q}) \cdot \dot{\mathbf{q}} \tag{7}$$

$$\ddot{\Delta}_i(\mathbf{q},\dot{\mathbf{q}},\ddot{\mathbf{q}}) = \begin{bmatrix} \dot{\mathbf{v}}_{C_i} \\ \dot{\omega}_i \end{bmatrix} = \mathbf{R}\Delta_i(\mathbf{q}) \cdot \ddot{\mathbf{q}} + {}_{\partial\mathbf{q}}\mathbf{R}\Delta_i(\mathbf{q}) \underset{1\backslash 3}{\otimes} \dot{\mathbf{q}} \cdot \dot{\mathbf{q}} \tag{8}$$

where $\mathbf{R}\Delta_i$ and ${}_{\partial\mathbf{q}}\mathbf{R}\Delta_i$ are compiled from the partial derivatives of Δ_i with respect to **q** [20].

For rigid body i the velocities that define its motion are stored in 6×1 matrix $\dot{\Delta}_{Ci} = \left[\mathbf{v}_{C_i}^{\backslash} \quad \omega_i^{\backslash}\right]^{\backslash}$ of the velocity of its centre of gravity C_i and the body angular velocity. For a flexible element with n nodes the $6.n\times1$ matrix of the velocities is $\dot{\Delta}_i = \left[\dot{\Delta}_{i,1}^{\backslash} \quad \dot{\Delta}_{i,2}^{\backslash} \quad \cdots \quad \dot{\Delta}_{i,n}^{\backslash}\right]^{\backslash}$, where $\dot{\Delta}_{i,j} = \left[\mathbf{v}_{i,j}^{\backslash} \quad \omega_{i,j}^{\backslash}\right]^{\backslash}$, $j = 1,2,...,n$ are 6×1 matrices of the node velocities. The Newton – Euler equations define the inertia forces and moments loading the body. For rigid body i the 6×1 matrix of the inertia forces and moments in the centre of gravity and relative to the inertial frame are:

$$\mathbf{F}_{C_i} = \mathbf{M}_{C_i} \cdot \ddot{\Delta}_{C_i} + \begin{bmatrix} {}^{3,3}\mathbf{0} & {}^{3,3}\mathbf{0} \\ {}^{3,3}\mathbf{0} & \omega_i^{\times} \end{bmatrix} \cdot \mathbf{M}_{C_i} \cdot \begin{bmatrix} {}^{3}\mathbf{0} \\ \omega_i \end{bmatrix} \tag{9}$$

where

$$\mathbf{M}_{C_i} = \begin{bmatrix} diag(m_i, m_i, m_i) & {}^{3,3}\mathbf{0} \\ {}^{3,3}\mathbf{0} & \tau_i \cdot \underline{\mathbf{J}}_i \cdot \tau_i^{\backslash} \end{bmatrix} \tag{10}$$

and m_i, $\underline{\mathbf{J}}_i$ are the body mass and inertia tensor, respectively. The underlined notations point out reference to body fixed coordinate system. In [20] dense (no zero elements) 6×6 mass matrices are regarded. These matrices are computed in cases of mass reduction, as it is for the finite element discretization, for which the node translations and rotations are dependent. For such matrices the generalized Newton – Euler equations [20] are:

$$\mathbf{F}_{C_i} = \mathbf{M}_{C_i} \cdot \ddot{\Delta}_{C_i} + \begin{bmatrix} \omega_i^{\times} & {}^{3,3}\mathbf{0} \\ \mathbf{v}_i^{\times} & \omega_i^{\times} \end{bmatrix} \cdot \mathbf{M}_{C_i} \cdot \begin{bmatrix} \mathbf{v}_{C_i} \\ \omega_i \end{bmatrix} - \mathbf{M}_{C_i} \cdot \begin{bmatrix} \omega_i^{\times} \cdot \mathbf{v}_{C_i} \\ {}^{3}\mathbf{0} \end{bmatrix} \tag{11}$$

The mass matrix of a flexible element i with n nodes is $6.n\times6.n$ symmetric positive defined dense matrix $\underline{\mathbf{M}}_i$. The mass matrices are computed assuming the equivalence of the kinetic energy of the deformable particles to the energy of the masses reduced to the nodes. If small relative deflections within the elements are assumed, these matrices are considered constant. The mass reduction is implemented on the basis of polynomial approximation of the beam deflections [21]. The kinetic energy of such an element is:

$$EK_i = \frac{1}{2}\dot{\Delta}_i^{\backslash} \cdot \tau\Delta_i \cdot \underline{\mathbf{M}}_i \cdot \tau\Delta_i^{\backslash} \cdot \dot{\Delta}_i = \frac{1}{2}\dot{\Delta}_i^{\backslash} \cdot \mathbf{M}_i \cdot \dot{\Delta}_i \tag{12}$$

where $\tau\Delta_i = diag(\tau_i, \tau_i, ..., \tau_i)$ is $6.n\times6.n$ coordinate system transformation matrix for flexible element i; $\mathbf{M}_i$ is $6.n\times6.n$ element mass matrix relative to inertial reference frame. The inertia forces ($6.n\times1$ matrix $\mathbf{F}_i$) in the nodes of flexible element i with n nodes are defined as follows:

$$\mathbf{F}_i = \mathbf{M}_i \cdot \ddot{\Delta}_i + \dot{\Delta}_i^{\otimes} \cdot \mathbf{M}_i \cdot \dot{\Delta}_i - \mathbf{M}_i \cdot \dot{\Theta}_i \tag{13}$$

where, $\dot{\Delta}_i^{\otimes} = diag\left(\dot{\Delta}_{i,1}^{\otimes}, \dot{\Delta}_{i,2}^{\otimes}, \cdots, \dot{\Delta}_{i,n}^{\otimes}\right)$; $\dot{\Delta}_{i,j}^{\otimes} = \begin{bmatrix} \omega_{i,j}^{\times} & {}^{3,3}\mathbf{0} \\ \mathbf{v}_{i,j}^{\times} & \omega_{i,j}^{\times} \end{bmatrix}$ is generalized skew-symmetric matrix of the linear and angular velocities of node j from element i; $\dot{\Theta}_i = \left[\dot{\Theta}_{i,1}^{\backslash} \quad \dot{\Theta}_{i,2}^{\backslash} \quad \cdots \quad \dot{\Theta}_{i,n}^{\backslash}\right]^{\backslash}$; $\dot{\Theta}_{i,j} = \begin{bmatrix} \omega_{i,j}^{\times} \cdot \mathbf{v}_{i,j} \\ {}^{3}\mathbf{0} \end{bmatrix}$.

The nodes of flexible elements achieve large displacements with respect to the inertial reference frame, while the deflections relative to the element coordinate system are small. In order the finite element stiffness matrices to be correctly applied the elastic forces loading the nodes are to be computed using only the small relative deflections of the nodes relative to the element.

The small deflections of element i with n nodes are compiled, similarly to the velocities in Sec. 4.1, in a $6.n\times 1$ matrix $\underline{\Delta}_i = \left[\underline{\Delta}^{\backslash}_{i,1} \quad \underline{\Delta}^{\backslash}_{i,2} \quad \cdots \quad \underline{\Delta}^{\backslash}_{i,n}\right]^{\backslash}$. The element stiffness properties are presented by $6.n\times 6.n$ sized stiffness $\underline{\mathbf{K}}_i$. For the finite elements the matrix $\underline{\mathbf{K}}_i$ is transformed to the absolute reference frame to compile the stiffness matrix $\mathbf{K}_i$. The elastic forces $\mathbf{S}_i = \left[\mathbf{S}^{\backslash}_{i,1} \quad \mathbf{S}^{\backslash}_{i,2} \quad \cdots \quad \mathbf{S}^{\backslash}_{i,n}\right]$ with respect to the absolute reference frame are computed using the relation $\mathbf{S}_i = -\mathbf{K}_i \cdot \Delta_i$. In a similar way the elastic forces relative to the moving element coordinate system are calculated taking into account the regulations for selection the reference coordinate systems of the flexible elements [22]. For example, the elastic forces in the beam element coordinate system are calculated by the well known stiffness matrix using the relation [20]:

$$\underline{\mathbf{S}}_i == -\underline{\mathbf{K}}_i \cdot \underline{\Delta}_i = -\underline{\mathbf{K}}_i \cdot \left[{}^{1,6}\mathbf{0} \quad \underline{\Delta}^{\backslash}_{i,2} \quad \cdots \quad \underline{\Delta}^{\backslash}_{i,n}\right] \qquad (14)$$

The process of computation of the node elastic forces goes trough the following steps:

- transformation of the coordinate systems of the nodes relative to the element coordinate systems;
- computation of the small relative node deflections;
- computation of the elastic forces in the nodes;
- transformation of the elastic forces to the absolute frame.

The final form of the dynamic equations with l rigid bodies, m flexible elements and n external forces is derived summing up the reduced inertia forces (Eqs. 11, 13) for all rigid and flexible objects, as well as, for all reduced external forces including the elastic forces (with common notation $\mathbf{G}_{M_i}$, $i = 1, 2, \ldots, n$), i. e.:

$$\sum_{i=1}^{l}\left[\mathbf{R}\Delta^{\backslash}_{C_i} \cdot \mathbf{M}_{C_i} \cdot \mathbf{R}\Delta_{C_i} \cdot \ddot{\mathbf{q}} + \mathbf{R}\Delta^{\backslash}_{C_i} \cdot \mathbf{F}_{C_i}(\dot{\mathbf{q}},\omega_i)\right] + \sum_{i=1}^{m}\left[\mathbf{R}\Delta^{\backslash}_{i} \cdot \mathbf{M}_{i} \cdot \mathbf{R}\Delta_{i} \cdot \ddot{\mathbf{q}} + \mathbf{R}\Delta^{\backslash}_{i} \cdot \mathbf{F}_{i}(\dot{\mathbf{q}},\dot{\Delta}_i)\right] - \sum_{i=1}^{n}\mathbf{R}\Delta^{\backslash}_{M_i} \cdot \mathbf{G}_{M_i} = {}^{g}\mathbf{0} \qquad (15)$$

Eg. 15 is $g\times 1$ linear system of ordinary differential equations for the generalized accelerations.

IV. Example

An example of application of the approach proposed to motion simulation of the six-link mechanism in Fig. 5 in case of singular configuration with no initial velocity is presented. The mechanism is of three chains and is decomposed of four branches. The kinematic scheme and elastic forces in the nodes of flexible element, link 3, ($f_{x3'}$, $f_{y3'}$, $m_{3'}$, $m_{3''}$, $f_{x3''}$, $f_{y3''}$) are presented in Fig. 7. Singular configuration for this mechanism is when the directrixes of the link 2, 3 (nodes 3'-3") and 5 (nodes 5'-5") are crossing in a common point. For this case the Jacobean matrix, as well as, the mass-matrix of the dynamic equations are singular.

Using the approach proposed no kinematic constraints that describe the connectivity of nodes 3' – 3" (of link 3) and 5' – 5" (of link 5) are applied. These constraints are the reason for the singularity and it is avoided substituting them by external elastic forces. On every step the relative position of the coordinate systems of links 3' – 3" and 5' – 5" is estimated and the small relative node deflections are computed. The elastic forces are calculated (Eq. 14) and used as external forces in the dynamic equations.

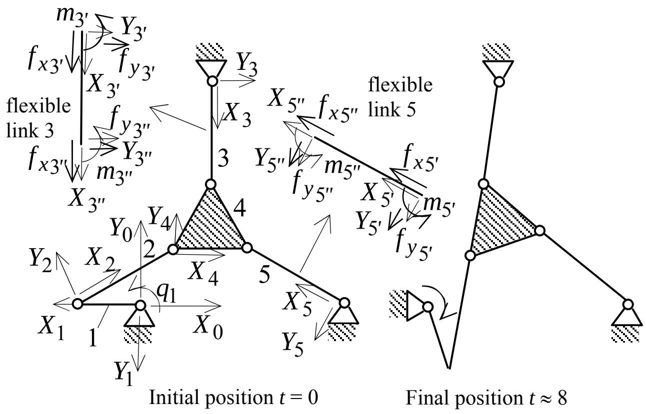

Fig. 7. Six link mechanism in its initial and final configuration

The kinematic parameters, the mass and stiffness properties of mechanism links are as follows (all measures are in SI UNITS): ternary link – size $1\times1\times1$, mass $m_4 = 3$, inertia moment $J_4 = 0.3$; link 1 - length $l_1 = 1$, $m_1 = 1$, $J_1 = 0.1$; link 2 – $l_2 = 2$, $m_2 = 2$, $J_2 = 0.2$; flexible links - $l_3 = l_5 = 2$, mass density $\rho = 3000$, modulus of elasticity $E = 0.7\times10^{11}$, cross section area $A = 4\times10^{-4}$, second moment of area $I_z = 2\times10^{-7}$. The initial configuration of the mechanism is defined by the coordinates $q_i, i = 1,3,5$ as it is shown in Fig. 7, i.e.: $q_1 = \pi$; $q_3 = -\pi/2$; $q_5 = 5\pi/6$. The prescribed motion is realized as a reonomic constraint for q_1, i.e.: $q_1 = \pi.cos\left(\frac{t}{4}\right)$ for $0 \le t \le 4$; $\dot{q}_1 = const$ for $t > 4$.

The time histories of the mechanism motion, links 1, 3 and 5, are presented in Fig. 8. The time histories of the elastic longitudinal forces in links 3 and 5 are shown in Fig. 9 and 10. The longitudinal elastic forces that arise as a result of the compulsive motion of the crank 1 do not cause significant influence of the transfer functions q_3 and q_5. The integration process starts from the initial singular configuration and stops when the mechanism reaches another singular configuration at $t \approx 8$. It could be seen that longitudinal elastic forces are exaggerated in the initial stage for going out from the singular configuration. These forces become higher at the end of motion since additional inertia forces appear and become extremely high at the interruption of the integration process.

The example demonstrates the applicability of the numerical algorithm. Future investigations will include damping and friction forces. Numerical integration methods for suppression of the high order vibrations will be applied and investigated.

V. Conclusions

An approach to simulation of rigid and flexible multibody system is proposed for which the kinematic constraints are substituted by elastic forces in the flexible links. The method provides general solution for every kind of closed chains including over-constrained mechanisms and singular configurations.

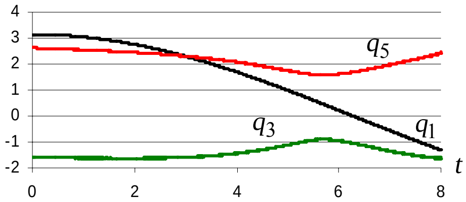

Fig. 8. Time history of the characteristics of motion of links 1, 3, 5

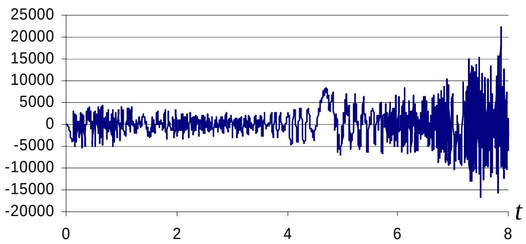

Fig. 9. Time history of the longitudinal elastic forces in link 3

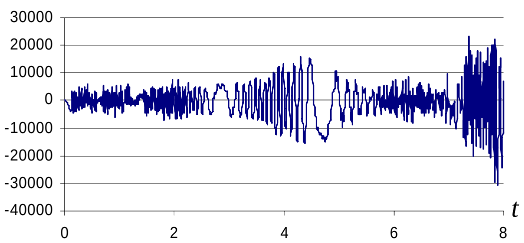

Fig. 10. Time history of the longitudinal elastic forces in link 5

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References

[1] Schiehlen W., (ed.). *Advanced Multibody System Dynamics. Solid Mechanics and its Application*. Kluwer Academic Publishers, Dordrecht, 1993.

[2] Haug E. J., *Computer-aided Kinematics and Dynamics of Mechanical Systems*. Allyn and Bacon, Boston, 1989.

[3] Shabana A. A., *Dynamics of Multibody Systems*. John Wiley & Sons, New York, 1989.

[4] García de Jalón, J. and Bayo E, *Kinematic and Dynamic Simulation of Multibody Systems. The Real-Time Challenge. Springer-Verlag,* New York, 1993.

[5] Angeles J. and Kecskemethy A., *Kinematics and Dynamics of Multibody Systems*. Springer-Verlag, Berlin, 1995.

[6] Bennet G. T., A new mechanism, *Engineering*, London, 76: 777 – 778, 1903.

[7] Gear C. W., The simultaneous numerical solution of differential-algebraic equations. *IEEE Trans. Circuit Theory*, CT-18:89—95, 1971.

[8] Baumgarte J., Stabilization of Constraints and Integrals of Motion. *Computer Methods in Applied Mechanics and Engineering*, (1):1-16, 1972.

[9] Kurdila A. J. and Narcovich F. J., Sufficient conditions for penalty formulation method in analytical dynamics. *Computational Mechanics*, 12:81 – 96, 1993

[10] Bayo E. and Ledesma R., Augmented Lagrangian and mass-orthogonal projection methods for constrained multibody dynamics. *Nonlinear Dynamics*, 9:113—130, 1996.

[11] Blajer W., Schiehlen W., and Schirm W.. A projective criterion to the coordinate partitioning method for multibody dynamics. *Applied Mechanics*, 64:86—98, 1994.

[12] Blajer W., A geometric unification of constrained system dynamics. *Multibody System Dynamcis*, 1:3—21, 1997.

[13] Wehage R. A. and Haug E. J., Generalized coordinate partitioning of dimension reduction in analysis of constrained dynamic systems. *ASME Journal of Mechanical Design*, 104:247—255, 1982.

[14] Bayo E. and Garcia de Jalon J.. A modified Lagrangian formulation for the dynamic analysis of constrained mechanical systems. *Computer methods in applied mechanics and engineering*, 71:183—195, 1988.

[15] J. Cuadrado J., Cardenal, and Bayo E., Modeling and solution methods for efficient real-time simulation of multibody dynamics. *Multibody System Dynamics*, 1:259—280, 1997.

[16] Arabian A. and Wu F., An improved formulation for constrained mechanical systems. *Multibody System Dynamics*, 2: 49-69, 1998.

[17] Eich-Soellner E. and Fuhrer C., *Numerical Methods in Multibody Dynamics*, B.G. Teubner, Stuttgart, 1998.

[18] Aghili F. and Piedboeuf J.-C., Simulation of motion of constrained multibody systems based on projection operator, *Multibody System Dynamics*, Kluwer Academic Publishers, 2002.

[19] Zahariev E. and McPhee J., Stabilization of multiple constraints using optimization and a pseudo-inverse matrix, *Mathematical and Computer Modeling of Dynamical Systems*, Swets & Zeitlinger Publishers, Netherlands, 9(4):423 – 441, 2003.

[20] Zahariev E., Generalized finite element approach to dynamics modeling of rigid and flexible systems, *Mechanics Based Design of Structures and Machines,* 34(1): 81-110, 2006.

[21] Zienkevich O. C., *T he Finite Elment Method*. McGraw-Hill, 1997.

[22] Zahariev E., Relative finite element coordinates in multibody ystem simulation, *Multibody System Dynamics*, Kluwer Academic Publishers, 7:51 – 77, 2002.