# REAL-TIME STATE OBSERVERS BASED ON MULTIBODY MODELS AND THE EXTENDED KALMAN FILTER

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## ABSTRACT

This work is a preliminary study on the use of the extended Kalman filter (EKF) for the state estimation of multibody systems. The observers based on the EKF are described by first-order differential equations, with independent, non-constrained coordinates. Therefore, it should be investigated how to formulate the equations of motion of the multibody systems so that efficient, robust and accurate observers can be derived, which can serve to develop advanced real-time applications. In the paper, two options are considered: a state-space reduction method and the penalty method. Both methods are tested on a four-bar mechanism with a linear spring-damper. The results enable to analyze the pros and cons of each method and provide clues for future research.

## 1. INTRODUCTION

The extended Kalman Filter (EKF) has been widely used, in combination with nonlinear dynamic models of systems, as state observer in several fields. The EKF for a nonlinear plant is in fact a simple linear Kalman Filter (KF) applied to the linearization of the plant around the trajectory given by the estimated state.

So, the main advantage of the EKF is that it inherits the optimality of the KF (optimality against state and measurement noise). Due to the linearization feature, this optimality becomes only local, not global. This implies the appearance of certain (usually large, but finite) “Domain of Attraction (DOA)”, that is, certain region of guaranteed convergence, which can be numerically precomputed (Delgado and Barreiro (2003)). In this way, the EKF provides robust observers for nonlinear systems.

In current practice, the EKF is combined with simplified dynamic models of the systems and elementary numerical integration schemes in order to streamline convergence and to achieve real-time performance of the computation process.

However, current state-of-the-art knowledge in multibody dynamics opens the possibility of considering complex multibody models in real-time state observer applications, as long as specialized schemes (that combine dynamic formulations and numerical integrators to produce robust and efficient algorithms) are employed (Cuadrado et al. (2004)). The advantage is that more information can be extracted from the model.

The EKF is typically formulated for first order nonlinear systems and non-constrained coordinates, in state-space form (ordinary differential equations, ODEs). Of course, the equations of a multibody system (differential algebraic equations, DAEs) can always be expressed in state-space form (minimum number of coordinates), and the corresponding second order equations can be converted into first order by just duplicating the number of variables.

However, practicability of the proposed strategy is limited by the ability of the resulting formalism to provide fast numerical execution and real-time performance. Therefore, different methods should be investigated so as to find the most efficient alternatives.

In the applications reported in the literature, the combination of the EKF with constrained DAE plants is usually addressed from the EKF point of view. That means adapting the KF rationale to the specific DAE problem. For example, Nikoukhah et al. (1990) show that the descriptor dynamics give rise to singular measurement noise covariance, and an extended maximum-likelihood method is applied. This same idea is followed in Chiang et al. (2002), where the constraint (unit quaternion norm) is treated as a pseudo-measurement. In De Geeter et al. (1997), the error from constraint linearization is treated in a separate step, after the EKF, increasing the computational complexity.

In this work, the solution to the combination of EKF and DAEs is approached from the DAE point of view. This is an advantage for complex multibody systems. As any observer runs in real-time a copy of the plant, the same techniques that are useful for modeling and fast simulation of complex multibody systems, will also be useful for implementing observers for such systems.

So, this is the main idea of our approach, based on intimately relate the EKF realization with efficient methods for dynamic formulation and fast execution of multibody systems. In particular, this work reports the EKF formal derivation in the case of a state-space reduction method and in the case of the penalty method.

Although the final objective of our project is to address complex multibody systems in industrial applications, in this first preliminary work a simple example is considered for clarity. This test example is a four-bar mechanism with a spring-damper element, so that conclusions based on this simple system can later serve to address larger and more complex systems.

Two computational versions of the mechanism are created: the first one represents the real “prototype”, while the second one plays the role of the “model”. In other to test the observer, the model is not an exact replica of the prototype, but differs in some physical or geometric parameters; also, the readings coming from sensors and actuators may be altered when passed to the model. The objective is that the model follows the motion of the prototype with the help of an EKF. Preliminary numerical results and practical discussions are presented at the end of the paper.

## 2. EKF OBSERVER

Consider the system (plant) given by:

$$\begin{aligned}\dot{\mathbf{x}} &= \mathfrak{A}\mathbf{x} + \mathfrak{B}\mathbf{u} + \boldsymbol{\delta} \\ \mathbf{y} &= \mathfrak{C}\mathbf{x} + \boldsymbol{\varepsilon}\end{aligned} \quad \textbf{(1)}$$

where $\mathbf{x}$ is the (unknown) state vector, and $\mathbf{u}$, $\mathbf{y}$ are the known input and measurement variables. The matrix coefficients $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ are also known and the equations are affected by state and measurement noises $\boldsymbol{\delta}$, $\boldsymbol{\varepsilon}$ with zero mean and given covariances $\boldsymbol{\Theta}$, $\boldsymbol{\Xi}$, respectively. Then, the classical *linear* Kalman filter (KF) observer is given by (Bryson and Ho (1975)):

$$\begin{aligned}\dot{\hat{\mathbf{x}}} &= \mathfrak{A}\hat{\mathbf{x}} + \mathfrak{B}\mathbf{u} + \mathfrak{K}\left(\mathbf{y} - \mathfrak{C}\hat{\mathbf{x}}\right) \\ \mathfrak{K} &= \mathbf{P}\mathfrak{C}^{\mathrm{T}}\boldsymbol{\Xi}^{-1} \\ \dot{\mathbf{P}} &= \mathfrak{A}\mathbf{P} + \mathbf{P}\mathfrak{A}^{\mathrm{T}} - \mathbf{P}\mathfrak{C}^{\mathrm{T}}\boldsymbol{\Xi}^{-1}\mathfrak{C}\mathbf{P} + \boldsymbol{\Theta}\end{aligned} \quad \textbf{(2)}$$

Notice that the state estimation has a prediction-correction structure, where the prediction is a copy of the plant ($\mathfrak{A}\mathbf{x}+\mathfrak{B}\mathbf{u}$) and the correction depends on the output error, affected by the Kalman gain $\mathfrak{K}$. The main feature of the KF is its optimality: it minimizes the covariance $\mathbf{P}$ of the state-estimation error.

When the plant is nonlinear:

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{f}\left(\mathbf{x},\mathbf{u}\right) + \boldsymbol{\delta} \\ \mathbf{y} &= \mathbf{h}\left(\mathbf{x}\right) + \boldsymbol{\varepsilon}\end{aligned} \quad \textbf{(3)}$$

Then, the *extended* Kalman filter (EKF) is given by (Bryson and Ho (1975)):

$$\begin{aligned}\dot{\hat{\mathbf{x}}} &= \mathbf{f}\left(\hat{\mathbf{x}},\mathbf{u}\right) + \mathfrak{K}\left(\mathbf{y} - \mathbf{h}\left(\hat{\mathbf{x}}\right)\right) \\ \mathfrak{K} &= \mathbf{P}\mathfrak{C}^{\mathrm{T}}\boldsymbol{\Xi}^{-1} \\ \dot{\mathbf{P}} &= \mathfrak{A}\mathbf{P} + \mathbf{P}\mathfrak{A}^{\mathrm{T}} - \mathbf{P}\mathfrak{C}^{\mathrm{T}}\boldsymbol{\Xi}^{-1}\mathfrak{C}\mathbf{P} + \boldsymbol{\Theta}\end{aligned} \quad \textbf{(4)}$$

But now the matrices $\mathfrak{A}$, $\mathfrak{C}$ are computed as the Jacobians of $\mathbf{f}$ and $\mathbf{h}$ with respect to the states, and are evaluated at the estimated trajectory (the true trajectory is not known).

The justification of the EKF is based on the principle of linearization. Actually, the EKF is nothing more that a linear KF applied to the linear plant (1) obtained by linearizing the true plant (3) around the estimated trajectory.

If the state estimation error is small, then the linearization error is small as well, and KF and EKF are actually the same. For this reason, it can be said that the EKF is *locally optimal*, or optimal for small errors.

But if for some reason (level of noise, disturbance, etc.), the estimation error becomes large, it might happen that the linearization error is so large that optimality is deteriorated and even stability is lost. If fact, EKF observers present, due to nonlinearity, a more or less large DOA that can be precomputed or estimated (Delgado and Barreiro (2003)).

## 3. MULTIBODY DYNAMICS

In its most basic form, the dynamics of a multibody system is described by the constrained Lagrangian equations:

$$\begin{aligned}\mathbf{M}\ddot{\mathbf{q}} + \boldsymbol{\Phi}_{\mathbf{q}}^{\mathrm{T}}\boldsymbol{\lambda} &= \mathbf{Q} \\ \boldsymbol{\Phi} &= \mathbf{0}\end{aligned} \quad \textbf{(5)}$$

where $\mathbf{M}$ is the positive semidefinite mass matrix, $\ddot{\mathbf{q}}$ the accelerations vector, $\boldsymbol{\Phi}$ the constraints vector, $\boldsymbol{\Phi}_{\mathbf{q}}$ the Jacobian matrix of the constraints, $\boldsymbol{\lambda}$ the Lagrange multipliers vector, and $\mathbf{Q}$ the applied forces vector, including the external actuation forces encoded in $\mathbf{u}$.

In order to adopt the form of the equations (3) required for application of the EKF, the second order system of equations (5) can be written as a first order one, just by doing $\mathbf{x}^T = \{\mathbf{q}^T \quad \mathbf{v}^T\}$ with $\mathbf{v} = \dot{\mathbf{q}}$,

$$\begin{aligned}\dot{\mathbf{q}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= \mathbf{M}^{-1}\left(\mathbf{Q} - \mathbf{\Phi}_{\mathbf{q}}^T\boldsymbol{\lambda}\right)\end{aligned} \quad \textbf{(6)}$$

or, more compactly,

$$\begin{Bmatrix}\dot{\mathbf{q}} \\ \dot{\mathbf{v}}\end{Bmatrix} = \begin{Bmatrix}\mathbf{v} \\ \mathbf{M}^{-1}\left(\mathbf{Q} - \mathbf{\Phi}_{\mathbf{q}}^T\boldsymbol{\lambda}\right)\end{Bmatrix} \Rightarrow \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \quad \textbf{(7)}$$

with the positions and velocities subject to the constraints at position and velocity level,

$$\begin{aligned}\mathbf{\Phi} &= \mathbf{0} \\ \mathbf{\Phi}_{\mathbf{q}}\mathbf{v} &= \mathbf{0}\end{aligned} \quad \textbf{(8)}$$

If $n_d$ is the number of dependent variables and $n_i$ is the number of degrees of freedom (independent variables) of the multibody system, the size of the problem is $2n_d$ (since the states are positions plus velocities).

In order to match (7) to (3), there are several problems. On the one hand, the Lagrange multipliers are unknowns, and the mass matrix is not always invertible. On the other hand, the formalism (3) does not consider constraints among the states.

One may resort to specific solutions for Kalman filtering of differential-algebraic equations (DAEs), as in the literature references discussed in the Introduction. However the explicit treatment of the model as a DAE would increase conceptual and computational complexity.

In our approach, two formulations that convert the DAE (5) into an ODE have been used: a state-space reduction method known as matrix-**R** method (Garcia de Jalon and Bayo (1994)) and the penalty method. The derivation of EKF observers for multibody systems based on the two mentioned methods is reported in the two following sections.

## 4. MATRIX-R FORMULATION

The main idea in this method (Garcia de Jalon and Bayo (1994)) is to obtain an ODE with dimension $n_i$ equal to the actual number of degrees of freedom, using a set **z** of independent coordinates. The starting point is to establish the following relation between velocities:

$$\dot{\mathbf{q}} = \mathbf{R}\dot{\mathbf{z}} \quad \textbf{(9)}$$

where **q** are all the $n_d$ dependent variables and **z** is a set of $n_i$ independent variables. Such relation (9) can always be found, for instance, taking the derivative of the restrictions, $\mathbf{\Phi}_{\mathbf{q}}\dot{\mathbf{q}} = \mathbf{0}$, and splitting all the velocities in two subsets, so that one subset of velocities can be written as a function of the other subset. Once (9) is obtained, it follows that

$$\ddot{\mathbf{q}} = \mathbf{R}\ddot{\mathbf{z}} + \dot{\mathbf{R}}\dot{\mathbf{z}} \quad \textbf{(10)}$$

Going back to (5), premultiplying by the transpose of **R**, and having in mind that $\mathbf{\Phi}_{\mathbf{q}}\mathbf{R} = \mathbf{0}$, one reaches to,

$$\ddot{\mathbf{z}} = \left(\mathbf{R}^T\mathbf{M}\mathbf{R}\right)^{-1}\left[\mathbf{R}^T\left(\mathbf{Q} - \mathbf{M}\dot{\mathbf{R}}\dot{\mathbf{z}}\right)\right] = \bar{\mathbf{M}}^{-1}\bar{\mathbf{Q}} \quad \textbf{(11)}$$

which implicitly defines the corrected mass matrix $\bar{\mathbf{M}}$ and the corrected vector of generalized forces $\bar{\mathbf{Q}}$.

So, the result is that the DAE (5) in the dependent variables has been converted into the ODE (11) expressed in independent variables.

The main advantage of the matrix-**R** method is the reduction of the number of equations, at the expense of having to compute, at each instant, **R** and the dependent states as functions of the independent ones. It also requires the effort of managing the redundancy in restrictions and the changes in the representative set of velocities.

If now the states are defined as $\mathbf{x}^T = \{\mathbf{z}^T \quad \mathbf{w}^T\}$, with $\mathbf{w} = \dot{\mathbf{z}}$, the following equations can be written,

$$\begin{aligned}\dot{\mathbf{z}} &= \mathbf{w} \\ \dot{\mathbf{w}} &= \left(\mathbf{R}^T\mathbf{M}\mathbf{R}\right)^{-1}\left[\mathbf{R}^T\left(\mathbf{Q} - \mathbf{M}\dot{\mathbf{R}}\mathbf{w}\right)\right] = \bar{\mathbf{M}}^{-1}\bar{\mathbf{Q}}\end{aligned} \quad \textbf{(12)}$$

or, more compactly,

$$\begin{Bmatrix}\dot{\mathbf{z}} \\ \dot{\mathbf{w}}\end{Bmatrix} = \begin{Bmatrix}\mathbf{w} \\ \bar{\mathbf{M}}^{-1}\bar{\mathbf{Q}}\end{Bmatrix} \Rightarrow \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \quad \textbf{(13)}$$

These equations perfectly match to (3) and, therefore, the EKF in (4) can be straightforwardly applied. In particular, the state-space matrix is obtained as the linearization (evaluated at the estimated trajectory):

$$\mathfrak{A} = \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}} = \begin{bmatrix}\mathbf{0} & \mathbf{I} \\ \dfrac{\partial\left(\bar{\mathbf{M}}^{-1}\bar{\mathbf{Q}}\right)}{\partial \mathbf{z}} & \dfrac{\partial\left(\bar{\mathbf{M}}^{-1}\bar{\mathbf{Q}}\right)}{\partial \mathbf{w}}\end{bmatrix} \quad \textbf{(14)}$$

which can be approximated as,

$$\mathfrak{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathfrak{A}_{21} & \mathfrak{A}_{22} \end{bmatrix}$$
$$\mathfrak{A}_{21} = -\bar{\mathbf{M}}^{-1}\mathbf{R}^{\mathrm{T}}\left(\mathbf{KR} + 2\mathbf{M}\mathbf{R}_{\mathbf{q}}\mathbf{R}\dot{\mathbf{w}}\right) \quad \textbf{(15)}$$
$$\mathfrak{A}_{22} = -\bar{\mathbf{M}}^{-1}\mathbf{R}^{\mathrm{T}}\left(\mathbf{CR} + \mathbf{M}\dot{\mathbf{R}}\right)$$

where **K** is the stiffness matrix and **C** the damping matrix.

In this case, the size of the problem is $2n_i$ . Now, according to (4), the correction provided by the EKF must be included into the observer equations, leading to

$$\dot{\mathbf{z}} - \mathbf{w} + \mathfrak{K}_1\left(\mathbf{y} - \mathbf{y}_s\right) = \mathbf{0}$$
$$\bar{\mathbf{M}}\dot{\mathbf{w}} - \bar{\mathbf{Q}} + \bar{\mathbf{M}}\mathfrak{K}_2\left(\mathbf{y} - \mathbf{y}_s\right) = \mathbf{0} \quad \textbf{(16)}$$

where $\mathfrak{K}_1$ and $\mathfrak{K}_2$ are the upper and lower parts of the Kalman gain matrix $\mathfrak{K}$, and $\mathbf{y}_s$ are the outputs provided by the sensors.

Since real-time performance of the algorithms will be required by the final applications, the integration procedure is relevant in order to make the algorithm as efficient as possible. The implicit single-step trapezoidal rule has been selected as integrator,

$$\dot{\mathbf{z}}_{n+1} = \frac{2}{\Delta t}\mathbf{z}_{n+1} - \left(\frac{2}{\Delta t}\mathbf{z}_n + \dot{\mathbf{z}}_n\right)$$
$$\dot{\mathbf{w}}_{n+1} = \frac{2}{\Delta t}\mathbf{w}_{n+1} - \left(\frac{2}{\Delta t}\mathbf{w}_n + \dot{\mathbf{w}}_n\right) \quad \textbf{(17)}$$

Now, (17) can be substituted into (16), thus leading to the nonlinear system of equations in the states,

$$\begin{cases} \mathbf{g}_1(\mathbf{x}_{n+1}) = \mathbf{0} \\ \mathbf{g}_2(\mathbf{x}_{n+1}) = \mathbf{0} \end{cases} \Rightarrow \quad \mathbf{g}(\mathbf{x}_{n+1}) = \mathbf{0} \quad \textbf{(18)}$$

This system can be iteratively solved by the Newton-Raphson iteration, the approximated tangent matrix being,

$$\frac{\partial \mathbf{g}}{\partial \mathbf{x}} = \begin{bmatrix} \frac{2}{\Delta t}\mathbf{I} & -\mathbf{I} \\ \mathbf{R}^{\mathrm{T}}\mathbf{KR} & \mathbf{R}^{\mathrm{T}}(\mathbf{CR} + \mathbf{M}\dot{\mathbf{R}}) + \frac{2}{\Delta t}\bar{\mathbf{M}} \end{bmatrix} + \begin{bmatrix} \mathfrak{K}_1\mathfrak{C}_1 & \mathfrak{K}_1\mathfrak{C}_2 \\ \bar{\mathbf{M}}\mathfrak{K}_2\mathfrak{C}_1 & \bar{\mathbf{M}}\mathfrak{K}_2\mathfrak{C}_2 \end{bmatrix} \quad \textbf{(19)}$$

where $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are the upper and lower parts of the output Jacobian matrix $\mathfrak{C}$.

## 5. PENALTY FORMULATION

The basic idea in the penalty method is to postulate that the constraining forces in (5) are proportional to the violation of the restrictions. In particular, the Lagrange multipliers are chosen in the form (Garcia de Jalon and Bayo (1994)):

$$\boldsymbol{\lambda} = \alpha\left(\ddot{\boldsymbol{\Phi}} + 2\zeta\omega\dot{\boldsymbol{\Phi}} + \omega^2\boldsymbol{\Phi}\right) \quad \textbf{(20)}$$

where $\alpha$ is the penalty factor, usually fixed to a very large value, $10^7$ or more. Notice that the combination of the constraint function and their derivatives takes the form of a second order oscillating system with damping coefficient and natural frequency usually chosen as $\zeta$=1, $\omega$=10. So, the rigid constraints in the DAE (5) can be converted into non-rigid constraints in an ODE:

$$\ddot{\mathbf{q}} = \left(\mathbf{M} + \boldsymbol{\Phi}_{\mathbf{q}}^{\mathrm{T}}\alpha\boldsymbol{\Phi}_{\mathbf{q}}\right)^{-1}\left[\mathbf{Q} - \boldsymbol{\Phi}_{\mathbf{q}}^{\mathrm{T}}\alpha\left(\dot{\boldsymbol{\Phi}}_{\mathbf{q}}\dot{\mathbf{q}} + 2\zeta\omega\dot{\boldsymbol{\Phi}} + \omega^2\boldsymbol{\Phi}\right)\right] \quad \textbf{(21)}$$

However, due to the very large value of $\alpha$, it can be shown that this is equivalent to representing the constraints by springs of large stiffness, dampers of large friction coefficient and masses of large inertia. In this way, the constraints can actually be violated, but only in a very small amount, enough for representing de DAE (5) as the ODE (21) with negligible approximation errors.

Compared to the matrix-**R** method (11), the equation (21) has the drawback that the number of variables is larger: it is equal to the total number of dependent variables, $n_d$ . However, this method has the advantage that (21) can be directly integrated as an ODE and it is not necessary to solve at each time instant the problems of passing from independent to dependent states and related problems mentioned in the previous section. Furthermore, the corrected mass matrix (the inverted matrix in (21)) is invertible, even if **M** is only positive semidefinite.

If now the states are defined as $\mathbf{x}^{\mathrm{T}} = \left\{\mathbf{q}^{\mathrm{T}} \;\; \mathbf{v}^{\mathrm{T}}\right\}$, with $\mathbf{v} = \dot{\mathbf{q}}$ , the following equations can be written,

$$\dot{\mathbf{q}} = \mathbf{v}$$
$$\dot{\mathbf{v}} = (\mathbf{M} + \boldsymbol{\Phi}_{\mathbf{q}}^{\mathrm{T}}\alpha\boldsymbol{\Phi}_{\mathbf{q}})^{-1}[\mathbf{Q} - \quad \textbf{(22)}$$
$$\boldsymbol{\Phi}_{\mathbf{q}}^{\mathrm{T}}\alpha(\dot{\boldsymbol{\Phi}}_{\mathbf{q}}\mathbf{v} + 2\zeta\omega\dot{\boldsymbol{\Phi}} + \omega^2\boldsymbol{\Phi})] = \bar{\mathbf{M}}^{-1}\bar{\mathbf{Q}}$$

or, more compactly,

$$\begin{Bmatrix} \dot{\mathbf{q}} \\ \dot{\mathbf{v}} \end{Bmatrix} = \begin{Bmatrix} \mathbf{v} \\ \bar{\mathbf{M}}^{-1}\bar{\mathbf{Q}} \end{Bmatrix} \quad \Rightarrow \quad \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \tag{23}$$

Then, the EKF in (4) can be straightforwardly applied. In particular, the state-space matrix is obtained as the linearization (evaluated at the estimated trajectory):

$$\mathfrak{A} = \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \dfrac{\partial\left(\bar{\mathbf{M}}^{-1}\bar{\mathbf{Q}}\right)}{\partial \mathbf{q}} & \dfrac{\partial\left(\bar{\mathbf{M}}^{-1}\bar{\mathbf{Q}}\right)}{\partial \mathbf{v}} \end{bmatrix} \tag{24}$$

which can be approximated as,

$$\begin{gathered} \mathfrak{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathfrak{A}_{21} & \mathfrak{A}_{22} \end{bmatrix} \\ \mathfrak{A}_{21} = -\bar{\mathbf{M}}^{-1}\left(\mathbf{K} + \omega^2 \mathbf{\Phi}_{\mathbf{q}}^{\mathrm{T}} \alpha \mathbf{\Phi}_{\mathbf{q}} + 2\mathbf{\Phi}_{\mathbf{q}}^{\mathrm{T}} \alpha \mathbf{\Phi}_{\mathbf{qq}} \dot{\mathbf{v}}\right) \\ \mathfrak{A}_{22} = -\bar{\mathbf{M}}^{-1}\left[\mathbf{C} + \mathbf{\Phi}_{\mathbf{q}}^{\mathrm{T}} \alpha \left(\dot{\mathbf{\Phi}}_{\mathbf{q}} + 2\zeta\omega \mathbf{\Phi}_{\mathbf{q}}\right)\right] \end{gathered} \tag{25}$$

In this case, the size of the problem is $2n_d$. As the size of the problem increases, it affects particularly to the covariance matrix **P** (see (4)), with a number of entries proportional to the square of the size.

Now, according to (4), the correction provided by the EKF must be included into the observer equations, leading therefore to

$$\begin{gathered} \dot{\mathbf{q}} - \mathbf{v} + \mathfrak{K}_1\left(\mathbf{y} - \mathbf{y}_s\right) = \mathbf{0} \\ \bar{\mathbf{M}}\dot{\mathbf{v}} - \bar{\mathbf{Q}} + \bar{\mathbf{M}}\mathfrak{K}_2\left(\mathbf{y} - \mathbf{y}_s\right) = \mathbf{0} \end{gathered} \tag{26}$$

where $\mathfrak{K}_1$ and $\mathfrak{K}_2$ are the upper and lower parts of the Kalman gain matrix $\mathfrak{K}$, and $\mathbf{y}_s$ are the outputs provided by the sensors.

As for the previous formulation, the implicit single-step trapezoidal rule has been selected as integrator,

$$\begin{gathered} \dot{\mathbf{q}}_{n+1} = \frac{2}{\Delta t}\mathbf{q}_{n+1} - \left(\frac{2}{\Delta t}\mathbf{q}_n + \dot{\mathbf{q}}_n\right) \\ \dot{\mathbf{v}}_{n+1} = \frac{2}{\Delta t}\mathbf{v}_{n+1} - \left(\frac{2}{\Delta t}\mathbf{v}_n + \dot{\mathbf{v}}_n\right) \end{gathered} \tag{27}$$

Now, (27) can be substituted into (26), thus leading to the nonlinear system of equations in the states,

$$\begin{cases} \mathbf{g}_1(\mathbf{x}_{n+1}) = \mathbf{0} \\ \mathbf{g}_2(\mathbf{x}_{n+1}) = \mathbf{0} \end{cases} \quad \Rightarrow \quad \mathbf{g}(\mathbf{x}_{n+1}) = \mathbf{0} \tag{28}$$

This system can be iteratively solved by the Newton-Raphson iteration, the approximated tangent matrix being,

$$\begin{gathered} \frac{\partial \mathbf{g}}{\partial \mathbf{x}} = \begin{bmatrix} \dfrac{2}{\Delta t}\mathbf{I} & -\mathbf{I} \\ \mathbf{T}_{21} & \mathbf{T}_{22} \end{bmatrix} + \begin{bmatrix} \mathfrak{K}_1\mathfrak{C}_1 & \mathfrak{K}_1\mathfrak{C}_2 \\ \bar{\mathbf{M}}\mathfrak{K}_2\mathfrak{C}_1 & \bar{\mathbf{M}}\mathfrak{K}_2\mathfrak{C}_2 \end{bmatrix} \\ \mathbf{T}_{21} = \mathbf{K} + \omega^2 \mathbf{\Phi}_{\mathbf{q}}^{\mathrm{T}} \alpha \mathbf{\Phi}_{\mathbf{q}} \\ \mathbf{T}_{22} = \mathbf{C} + \mathbf{\Phi}_{\mathbf{q}}^{\mathrm{T}} \alpha \left(\dot{\mathbf{\Phi}}_{\mathbf{q}} + 2\zeta\omega \mathbf{\Phi}_{\mathbf{q}}\right) + \frac{2}{\Delta t}\bar{\mathbf{M}} \end{gathered} \tag{29}$$

where $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are the upper and lower parts of the output Jacobian matrix $\mathfrak{C}$.

## 6. EXAMPLE

The four-bar mechanism with a spring-damper element shown in Fig. 1 is chosen as example. Two computational versions of the mechanism are created: the first one represents the real "prototype", while the second one plays the role of the "model". A sensor in the prototype provides as a measurement $y=s$, the distance between point A and point 2, i.e. the ends of the spring-damper element.

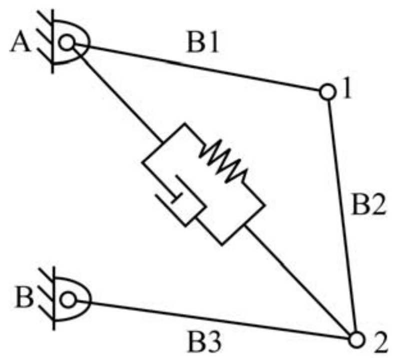


Figure 1. FOUR-BAR MECHANISM WITH SPRING-DAMPER ELEMENT.

The state variables in this example are the Cartesian coordinates of points 1 and 2, and the distance $s$:

$$\mathbf{q}^{\mathrm{T}} = \{x_1 \quad y_1 \quad x_2 \quad y_2 \quad s\} \tag{30}$$

The inertial matrix of the whole mechanism is computed by considering the contributions of the three 2D bars (Garcia de Jalon and Bayo (1994)) and the zero contribution of the spring-damper element (of negligible mass):

$$\mathbf{M}=\begin{bmatrix} \frac{m_{B1}+m_{B2}}{3} & 0 & \frac{m_{B2}}{6} & 0 & 0 \\ 0 & \frac{m_{B1}+m_{B2}}{3} & 0 & \frac{m_{B2}}{6} & 0 \\ \frac{m_{B2}}{6} & 0 & \frac{m_{B2}+m_{B3}}{3} & 0 & 0 \\ 0 & \frac{m_{B2}}{6} & 0 & \frac{m_{B2}+m_{B3}}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \mathbf{(31)}$$

Notice the fifth row and column of zeros, as there is no inertia associated to the spring-damper element.

The generalized force vector is obtained considering the gravity as the only force that acts over each bar. So, for the three bars it results (Garcia de Jalon and Bayo (1994)):

$$\mathbf{Q}=\begin{Bmatrix} 0 \\ -\frac{m_{B1}g}{2}-\frac{m_{B2}g}{2} \\ 0 \\ -\frac{m_{B2}g}{2}-\frac{m_{B3}g}{2} \\ -k\left(s-s_0\right)-c\dot{s} \end{Bmatrix} \quad \mathbf{(32)}$$

where $k$ is the elastic constant, $c$ the damping coefficient and $s_0$ the natural spring length.

Finally, the constraints vector is:

$$\mathbf{\Phi}=\begin{Bmatrix} \left(x_1-x_A\right)^2+\left(y_1-y_A\right)^2-L_{B1}^2 \\ \left(x_2-x_1\right)^2+\left(y_2-y_1\right)^2-L_{B2}^2 \\ \left(x_2-x_B\right)^2+\left(y_2-y_B\right)^2-L_{B3}^2 \\ \left(x_2-x_A\right)^2+\left(y_2-y_A\right)^2-s^2 \end{Bmatrix} \quad \mathbf{(33)}$$

The numerical values of the parameters are: masses $m_{B1}$=2 kg, $m_{B2}$=25 kg, $m_{B3}$=2 kg; bar lengths $L_{B1}$=0.9 m, $L_{B2}$=1 m, $L_{B3}$=1.05 m; natural spring length $s_0$=1.4 m; fixed points Cartesian coordinates $A$=(0,0), $B$=(0,–1); spring coefficient $k$=10000 N/m, and damping coefficient $c$=500 Ns/m.

Regarding the tuning of the EKF, the main parameters are the matrices Θ, Ξ. In principle, they have to represent the covariances of the zero-mean noises δ, ε in (1) or (3). However, in practice, this information is not always clearly known, so in fact the matrices **Θ**, Ξ are used as tuning parameters adjusted by trial-and-error experimental work.

It should be stressed that there is no perfect solution to the filtering problem, but rather there exist trade-offs between competing objectives. Typically, the solutions that provide fast convergence (of the estimations to the true values) are affected by higher levels of noise. If low levels of noise are desired, then the initial errors converge more slowly to zero. To facilitate the tuning, typically the matrices are postulated to be diagonal, **Θ**=diag( $\theta_i$ ), Ξ=diag( $\xi_i$ ).

After some trial-and-error test work under the simulation conditions to be detailed later, the EKF tuning is set to **Θ**=diag(10), Ξ=diag(0.01). The initial covariance value has been chosen to be **P**(0)=diag(diag( $p_i$ ),diag( $v_i$ )), to represent different initial uncertainties in positions and velocities. As it is supposed that model and prototype start from rest conditions, the initial uncertainty is zero in velocities and, let us say, 0.1 in positions, so that **P**(0)=diag(diag(0.1),diag(0)).

Regarding the simulation conditions, to show the recovering from different initial conditions, the real prototype starts at $s$(0)=1.80 m, while the virtual model (observer) starts at $s$(0)=1.85 m. In the same way, to evaluate the effect of noise and error in measurements, it is supposed that $y=s+\varepsilon$, with measurement noise $\varepsilon$ within the interval [–0.02,+0.02] m (2 cm), uniformly distributed. Finally, in order to check what happens with uncertain exogenous forces, the prototype runs under normal gravity, $g$=9.81 m/s$^2$, but the observer runs under $g$=8.81 m/s$^2$.

In these conditions, the history of the states $s$, $x_1$ and $\dot{x}_1$ (real vs. estimated through both matrix-**R** and penalty method) is plotted in Fig. 2, Fig. 3 and Fig. 4.

Furthermore, to address a more difficult situation, Fig. 5, Fig. 6 and Fig. 7 show the evolution of the same states under a harder noise level [–0.1,+0.1] m, with a null damping coefficient for the prototype and a value of $c$=100 Ns/m for the model.

All the simulation plots show a reasonable good performance level, with satisfactory convergence speed, robustness against parameter uncertainty, and noise filtering and attenuation.

Regarding the two multibody methods being compared, the plots show almost identical behavior for both of them (only, greater instability is perceived for the penalty method at some instants of Fig. 7). However, the following remarks must be done.

It has been said that, unlike the matrix-**R** method, the penalty method does not require the solution of the position and velocity problems at each function evaluation, which represents an advantage. Yet, if such problems are not solved, the observer based on the penalty method behaves worse than the one based on the matrix-**R** method. The reason is that, while the measured coordinate $s$ in perfectly followed by the observer, the remaining coordinates $x_1$, $y_1$, $x_2$, $y_2$ are not consistent with the distance $s$, since

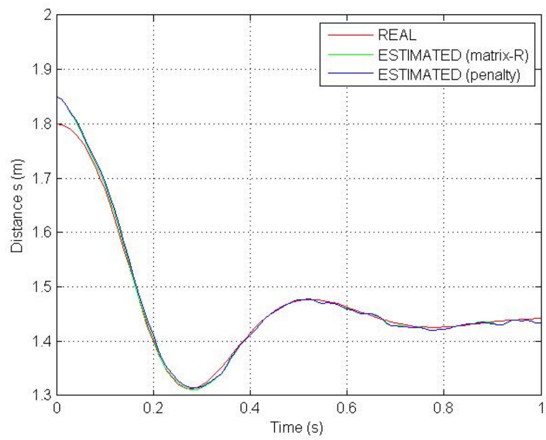


Figure 2. HISTORY OF STATE *s*.

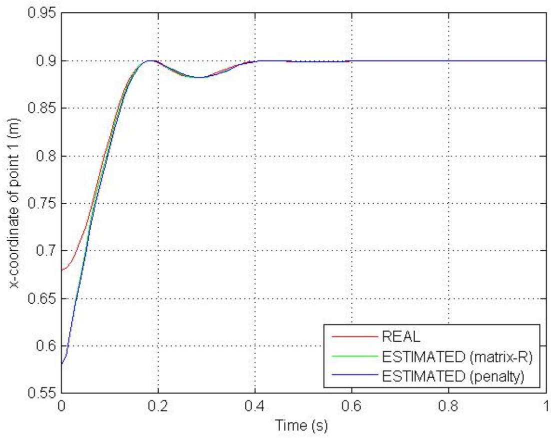


Figure 3. HISTORY OF STATE $x_1$.

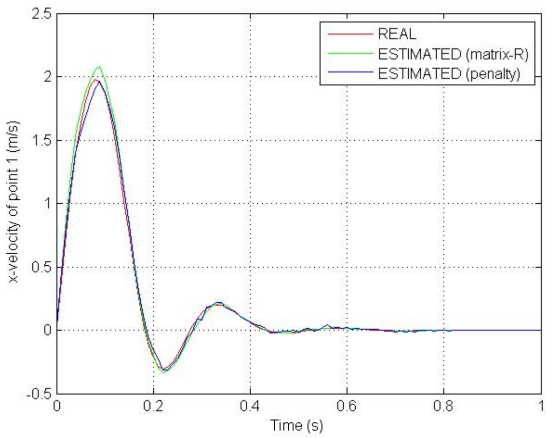


Figure 4. HISTORY OF STATE $\dot{x}_1$.

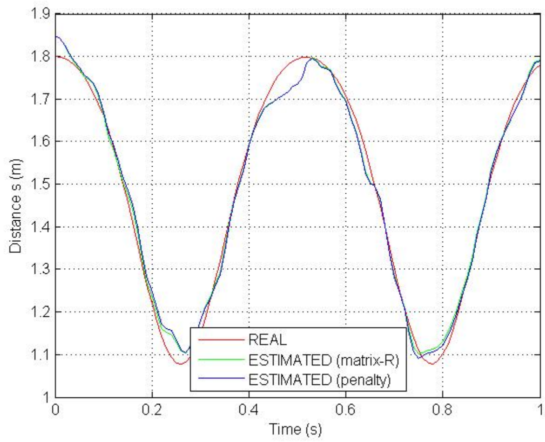


Figure 5. HISTORY OF STATE *s* (UNDAMPED PROTOTYPE).

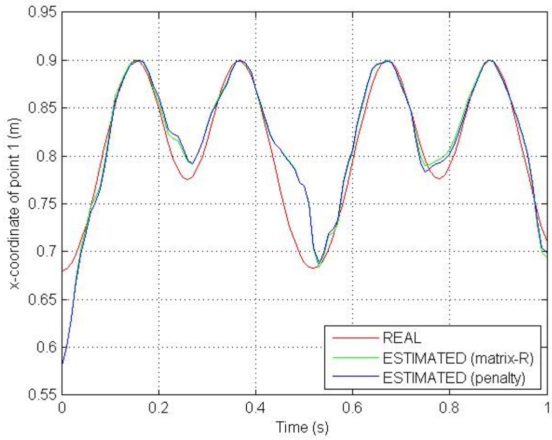


Figure 6. HISTORY OF STATE $x_1$ (UNDAMPED PROTOTYPE).

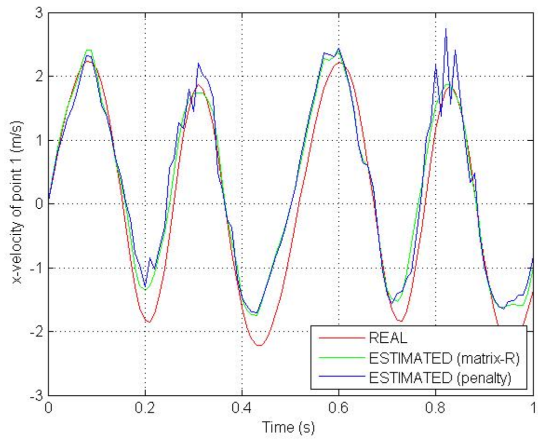


Figure 7. HISTORY OF STATE $\dot{x}_1$ (UNDAMPED PROTOTYPE).

the penalty terms are not capable of ensuring the constraint satisfaction under the large forces introduced by the EKF. And this happens no matter which the value of the penalty factor is.

The explanation to this phenomenon can be found by looking at the second equation in (26). It can be seen that an increment of the penalty factor increases the penalty forces which oppose to constraint violation but, at the same time, increases the value of the correction terms coming from the EKF. Therefore, increasing the value of the penalty factor in order to ensure constraint satisfaction is worthless. Consequently, when using the penalty method, the position and velocity problems must be solved at each function evaluation, so that constraints are fulfilled, and good results are obtained, at the cost of losing this advantage with respect to the matrix-**R** method.

The CPU-times required for both methods to run the two described simulations under Matlab environment are gathered in Table 1. The fixed time-step used for the numerical integration is, in all cases, $\Delta t$=0.01 s.

Table 1. CPU-TIMES (s).

| **Method** | **Case 1** | **Case 2** |
|---|---|---|
| Matrix-**R** | 0.11 | 0.13 |
| Penalty | 0.18 | 0.21 |

Therefore, it is clear that the matrix-**R** method is more efficient than the penalty method. Likely, this fact is due to the above-mentioned need of solving the position and velocity problems in the penalty method.

## 7. CONCLUSIONS

This work presents a study on the application of EKF observers to multibody systems. Although the numerical tests have been carried out, by the moment, on a simple four-bar example, the final objective is to implement the observers on complex multibody systems for advanced real-time applications.

So, having in mind complex multibody systems, the approach has been based on the idea that the same techniques which are efficient for efficient simulation of such complex systems, will be efficient as well for implementing the observers. Two methods have been chosen: the matrix-**R** formulation and the penalty formulation. The detailed development of the EKF observer for these methods has been presented.

The simulation tests show successful results in both cases. After a not very involved trial-and-error tuning, the EKF final observers are robust with respect to sensor noise, initial estimation errors and even different input forces. The matrix-**R** method has the advantages of leading to a lower problem size and perfectly matching to the EKF formalism. The penalty method was expected to have the advantage of not requiring the solution of the position and velocity problems at each function evaluation. However, it was found that constraint satisfaction was not achieved by the penalty terms and, therefore, the mentioned advantage was lost. Consequently, the matrix-**R** method showed to be more efficient.

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## REFERENCES

Bryson, A.E., Ho, Y.-Ch., *Applied Optimal Control. Optimization, Estimation and Control*, Wiley, New York, 1994.

Chiang, Y.T., Wang, L.S., Chang, F.R. Peng, M.R., Constrained filtering method for attitude determination using GPS and gyro, *IEE Proc. Radar Sonar Navig.*, Vol. 149, No. 5, pp. 258-264, 2002.

Cuadrado, J., Dopico, D., Naya, M.A., Gonzalez, M., Penalty, semi-recursive and hybrid methods for MBS real-time dynamics in the context of structural integrators, *Multibody System Dynamics*, Vol. 12, No. 2, pp. 117-132, 2004.

De Geeter, J., Van Brussel, H., De Schutter, J., A smoothly constrained Kalman filter, *IEEE Trans. on Pattern Analysis and Machine Intelligence*, Vol. 19, No. 10, pp. 1171-1177, 1997.

Delgado, E. and Barreiro, A., Sonar-based robot navigation using nonlinear robust observer, *Automatica*, Vol. 39, pp. 1195-1203, 2003.

Garcia de Jalon, J., Bayo, E., *Kinematic and Dynamic Simulation of Multibody Systems*, Springer-Verlag, New York, 1994.

Nikoukhah, R., Willsky, A.S., Levy, B.C., Kalman filtering and Riccati equations for descriptor systems, *Proceedings of the 29th Conference on Decision and Control*, Honolulu, Hawaii, 1990.