Proceedings of the ASME 2018 International Design Engineering Technical Conferences & Computers and Information in Engineering Conference
IDETC/CIE 2018
August 6-9, 2018, Québec City, QC, Canada

DETC2018-85203

MINIMAL REALIZATIONS FOR LINEAR MULTIBODY SYSTEM DYNAMICS

Bruce Minaker*, **Francisco González**†

* Mechanical, Automotive & Materials Engineering, University of Windsor
401 Sunset Avenue, Windsor, ON, Canada
bminaker@uwindsor.ca

†Laboratorio de Ingeniería Mecánica, University of A Coruña
Mendizábal s/n, 15403 Ferrol, Spain
Email: f.gonzalez@udc.es

ABSTRACT

Multibody dynamics formulations usually express the dynamics of a mechanical system as a set of highly nonlinear Ordinary Differential Equations (ODEs) or Differential Algebraic Equations (DAEs). However, the equations of motion thus obtained need to be linearized for their use in a number of applications, such as modal analysis, frequency response, or feedback control development. When representing the governing equations of any linear system, one of the relevant problems is the determination of the mathematically equivalent formulation of the smallest size, and equivalently, the lowest order. This work discusses the challenges associated with the relevant procedures, and proposes a method based on the Jordan form of the system of equations.

1 INTRODUCTION

When representing the governing equations of any linear system, one of the relevant problems is the determination of the mathematically equivalent formulation of the smallest size, and equivalently, the lowest order. This work relates specifically to the development of a multibody dynamics based vehicle motion simulation, based on the equations of motion generator code EoM, developed by the University of Windsor Vehicle Dynamics and Control research group, although the results would be equally applicable in any similar implementation. The EoM software is able to generate equations of motion for complex three dimensional multibody systems, but restricts the result to linear equations.

When generating the linearized equations of motion, many authors will choose to present them in the traditional linear second order form shown in Eqn. (1).

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{L}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f} \tag{1}$$

In this form, the matrices $\mathbf{M}$, $\mathbf{L}$, and $\mathbf{K}$ represent the mass, damping, and stiffness respectively, $\mathbf{x}$ is the vector of translational and rotational motions, and $\mathbf{f}$ is the vector of applied forces and moments. Another useful alternative is to prepare the equations in linear first order, or *state space* form, as shown in Eqn. (2).

$$\begin{Bmatrix} \dot{\mathbf{x}} \\ \mathbf{y} \end{Bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{Bmatrix} \mathbf{x} \\ \mathbf{u} \end{Bmatrix} \tag{2}$$

where vectors $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{u}$ represent the states of the system, the outputs, and the inputs, respectively. The state vector may be the translational and rotational displacements and velocities, but there are other possibilities. The $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ matrices are the system, input, output, and feed-through, respectively. The second order form can be easily reduced to state space form with standard mathematical manipulation.

*Address all correspondence to this author.

One of the interesting properties of the first order form of the equations is the concept of the *minimal realization*. The input/output relationship between $\mathbf{u}$ and $\mathbf{y}$ is expressed using a non-unique set of matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$. An intriguing feature is that not all sets of $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$, or *realizations*, need be of the same dimension. In fact, there is a theoretical lower limit, known as the *McMillan degree*, that denotes the minimum possible dimension of the square $\mathbf{A}$ matrix. A realization in which the dimension of the system matrix is matching the McMillan degree is known as a *minimal realization*, and the task of computing one, starting from an existing non-minimal realization is a well studied problem in linear systems analysis [1].

1.1 Controllability and Observability

In order to fully describe the minimization process, the concepts of *controllability* and *observability* must be introduced. If any of the modes that result from a modal analysis of the equations of motion lies perpendicular to the input forcing vector (a column of the input matrix), then that particular mode cannot be excited by the associated input. If any modes exist that cannot be excited by any of the inputs, then the system is said to be uncontrollable. Similarly, if a mode lies perpendicular to the measurement vector (a row of the output matrix), it cannot be detected. If any modes exist that cannot be detected by any of the outputs, the system is said to be unobservable. In a minimal system, no uncontrollable modes and no unobservable modes are present. The minimization procedure is based on the concept of eliminating any modes that are unobservable or uncontrollable. A system that is both controllable and observable is also minimal. A number of methods have been described in the literature to remove the unobserveable and uncontrollable modes.

2 IMPLEMENTATION

The procedures described above are evaluated in an open source code, acronymed as 'EoM', which has been developed to automatically generate the linearized equations of motion for a multibody system, particularly vehicles [2]. The equations produced by EoM are presented in descriptor state space form, as seen in Eqn. (3).

$$\begin{bmatrix} \mathbf{E} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{x}} \\ \mathbf{y} \end{Bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{Bmatrix} \mathbf{x} \\ \mathbf{u} \end{Bmatrix} \tag{3}$$

The equations generated by EoM are not initially minimal, until one of the procedures mentioned above is employed. Along with the state space matrices, the EoM software will optionally generate a set of indicators for comparison with results generated with other tools, e.g., the eigenvalues of the system, the frequency response, or the Hankel singular values. The source code for the EoM software is distributed under an open-source license, and is freely available online (on the popular source code sharing site github.com).

3 JORDAN FORM

This work explores the potential of an alternate approach based on modal identification, and the Jordan form $\mathbf{J}$ of the system matrix, shown in Eqn. (4).

$$\mathbf{J} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_q \end{bmatrix} \text{ where } \mathbf{J}_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix} \tag{4}$$

The matrix $\mathbf{T}$ represents a coordinate transformation, and λ_i represents the i^{th} eigenvalue of $\mathbf{A}$. The Jordan form of the system matrix is similar to the diagonal form that results from an eigen decomposition. The primary distinction of the Jordan form is in the case of repeated eigenvalues, and in particular the case where the eigenvectors fail to form a basis. The Jordan form utilizes the concept of *generalized eigenvectors* to complete the basis.

A diagonal system matrix is useful as it shows directly the contribution of each input to each mode, and the contribution of each mode to each output. Off-diagonal terms in the system matrix couple two state equations, and complicate the assessment of the the coupling of each input to each mode. The amount of off-diagonal coupling in the Jordan form of a general matrix varies, and is dependant on the number of repeated eigenvalues; it requires a distinction between the *algebraic multiplicity* and *geometric multiplicity* of the eigenvalues. In the case that the system equations represent the equations of motion of a multibody system, the form of the off-diagonal elements can be predicted, and used to simplify the determination of the contribution of each mode. Those modes that are shown to be non-contributing can be eliminated from the equations of motion.

ACKNOWLEDGMENT

The second author acknowledges the support of the Spanish Ministry of Economy through the Ramón y Cajal program, contract No. RYC-2016-20222.

REFERENCES

[1] B.C. Moore, "Principal Component Analysis in Linear Systems: Controllability, Observability, and Model Reduction," *IEEE Trans. on Auto. Control*, vol. AC-26, no. 1, pp. 17–32, 1981.

[2] Minaker, B., Rieveley, R., 2010. "Automatic Generation of the Non-holonomic Equations of Motion for Vehicle Stability Analysis", *Vehicle System Dynamics*, ***49*** (9), pp. 1043–1063.