Sensitivity analysis and optimization of the dynamics of multibody systems using analytical gradient based methods

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Doctoral thesis

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UNIVERSIDADE DA CORUÑA

Outline

1) Introduction and motivation
2) Dynamics of open-loop systems
3) Dynamics of closed-loop systems
4) Sensitivity analysis of unconstrained open-loop systems
5) Sensitivity analysis of closed-loop systems
6) MBSLIM implementation
7) Numerical experiments
8) Conclusions

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1) **Introduction and motivation**
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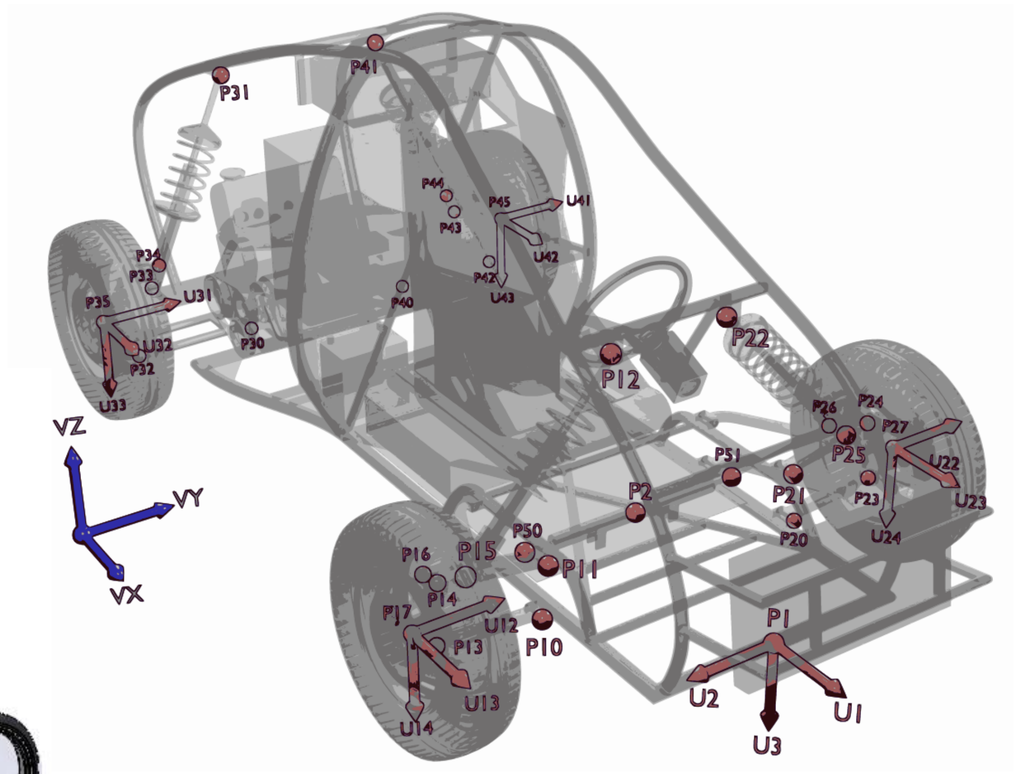
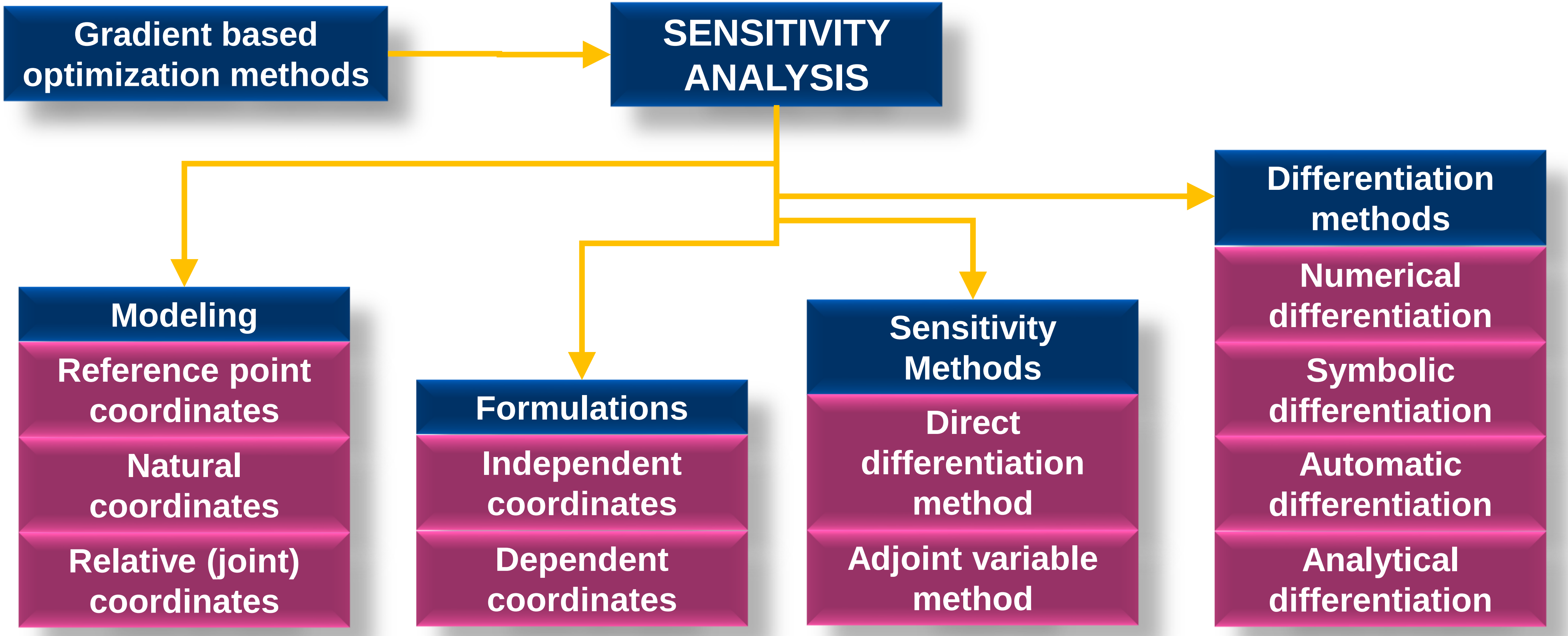
Introduction and motivation.

- **Goal:** Optimal control and design optimization.
- **Method:** Gradient based optimization.
- **Characteristics:** highly dependent on dynamics and sensitivity evaluations.
- **Solution:** accurate and highly efficient dynamic and sensitivity formulations by means of:
  - Joint coordinate models.
  - Analytical differentiation.
  - General dynamic and sensitivity formulations.
  - General and robust software.



State of the art.



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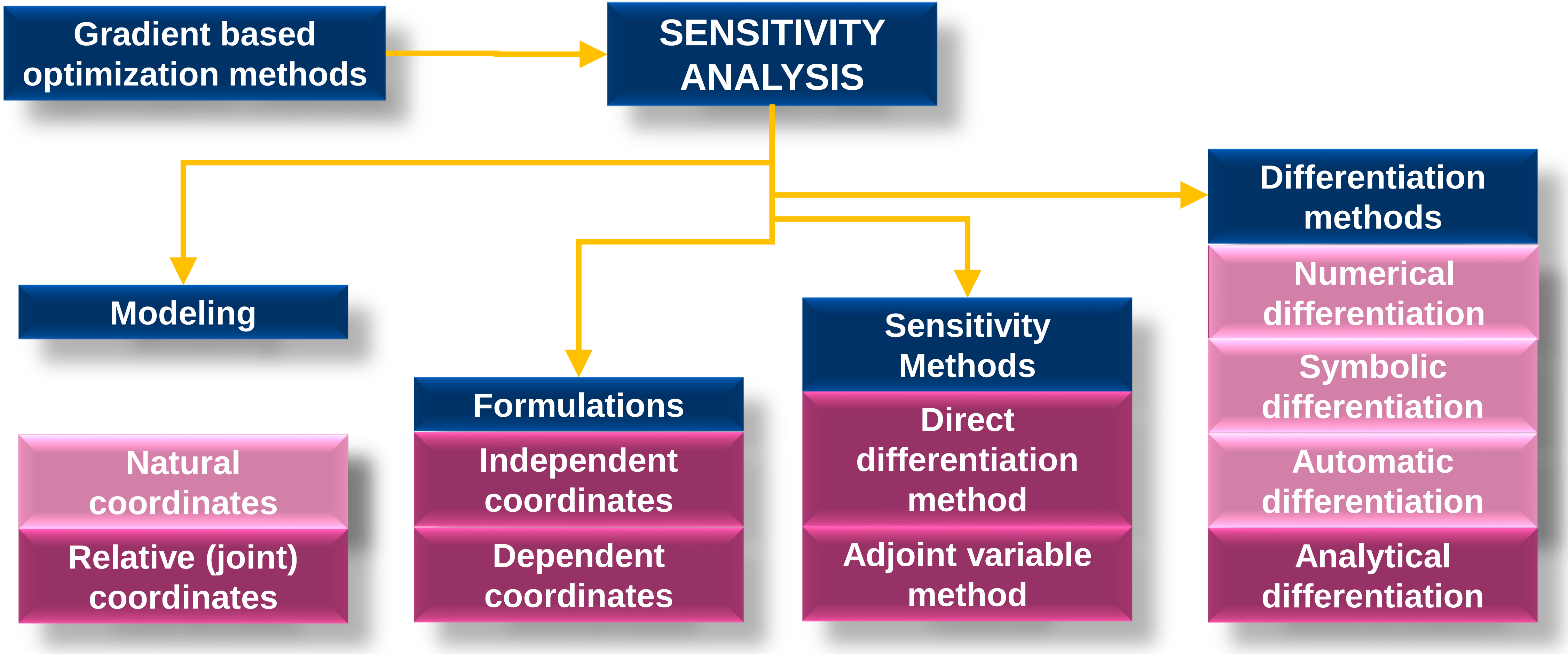

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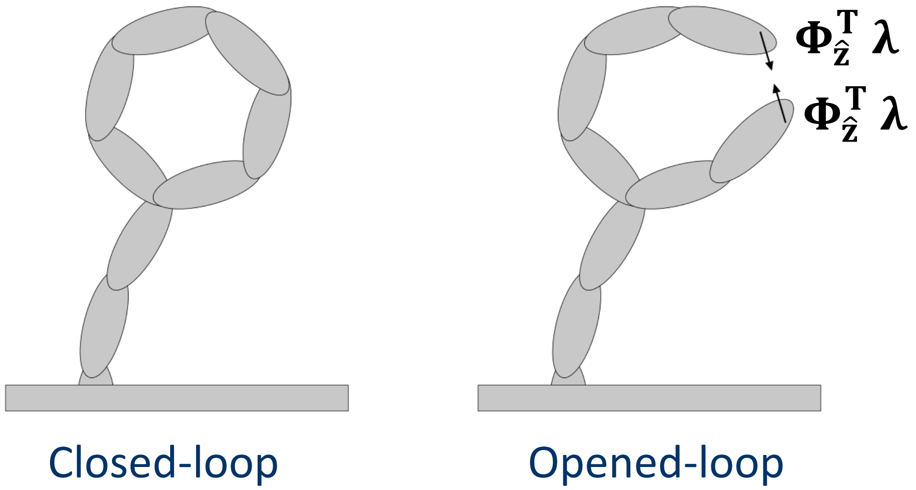
Joint coordinate models.

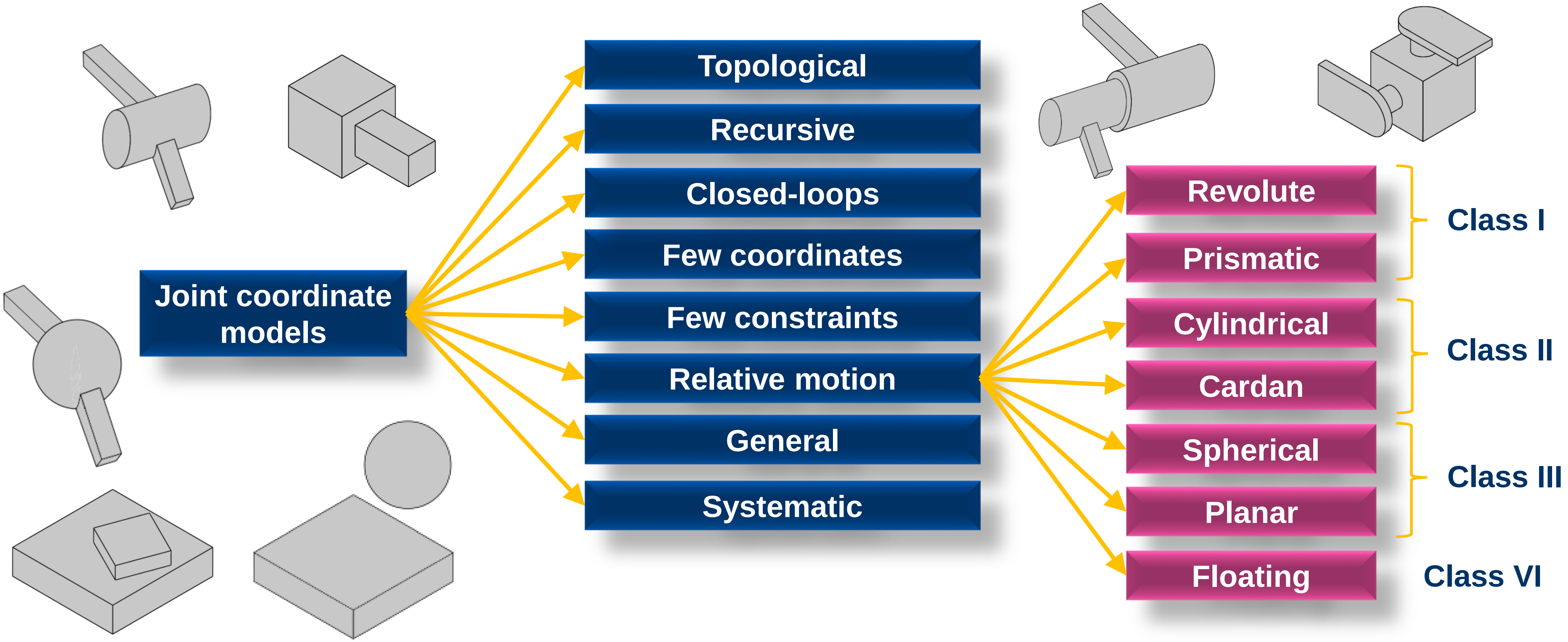
Joint coordinate models

- Topological
- Recursive
- Closed-loops



Dynamics of open-loop systems:

Joint coordinate models.



Recursive kinematic relations.

- Relative motion at velocity level:

$$\dot{\mathbf{r}}_i = \dot{\mathbf{r}}_i^{i-1} + \dot{\mathbf{r}}_i^{i,i-1}$$
$$\dot{\mathbf{r}}_i^{i-1} = \dot{\mathbf{r}}_{i-1} + \boldsymbol{\omega}_{i-1} \wedge \left(\mathbf{r}_i - \mathbf{r}_{i-1}\right)$$
$$\boldsymbol{\omega}_i = \boldsymbol{\omega}_{i-1} + \boldsymbol{\omega}_i^{i-1}$$

- In matrix form:

$$\begin{bmatrix} \dot{\mathbf{r}}_i \\ \boldsymbol{\omega}_i \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \tilde{\mathbf{r}}_{i-1} - \tilde{\mathbf{r}}_i \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{r}}_{i-1} \\ \boldsymbol{\omega}_{i-1} \end{bmatrix} + \begin{bmatrix} \dot{\mathbf{r}}_i^{i,i-1} \\ \boldsymbol{\omega}_i^{i-1} \end{bmatrix}$$

- Identifying terms:

$$\mathbf{V}_i = \begin{bmatrix} \dot{\mathbf{r}}_i \\ \boldsymbol{\omega}_i \end{bmatrix}, \quad \mathbf{V}_{i-1} = \begin{bmatrix} \dot{\mathbf{r}}_{i-1} \\ \boldsymbol{\omega}_{i-1} \end{bmatrix},$$

$$\mathbf{B}_i^v = \begin{bmatrix} \mathbf{I} & \tilde{\mathbf{r}}_{i-1} - \tilde{\mathbf{r}}_i \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \quad \mathbf{b}_i^v \dot{\mathbf{z}}_i = \begin{bmatrix} \dot{\mathbf{r}}_i^{i,i-1} \\ \boldsymbol{\omega}_i^{i-1} \end{bmatrix}.$$

- Relative motion at acceleration level:

$$\ddot{\mathbf{r}}_i = \ddot{\mathbf{r}}_i^{i-1} + \ddot{\mathbf{r}}_i^{i,i-1} + 2\boldsymbol{\omega}_{i-1} \wedge \dot{\mathbf{r}}_i^{i,i-1}$$
$$\ddot{\mathbf{r}}_i^{i-1} = \ddot{\mathbf{r}}_{i-1} + \dot{\boldsymbol{\omega}}_{i-1} \wedge \left(\mathbf{r}_i - \mathbf{r}_{i-1}\right) + \boldsymbol{\omega}_{i-1} \wedge \left[\boldsymbol{\omega}_{i-1} \wedge \left(\mathbf{r}_i - \mathbf{r}_{i-1}\right)\right]$$
$$\dot{\boldsymbol{\omega}}_i = \dot{\boldsymbol{\omega}}_{i-1} + \boldsymbol{\alpha}_i^{i-1} + \boldsymbol{\omega}_{i-1} \wedge \boldsymbol{\omega}_i^{i-1}$$

- In matrix form:

$$\begin{bmatrix} \ddot{\mathbf{r}}_i \\ \dot{\boldsymbol{\omega}}_i \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \tilde{\mathbf{r}}_{i-1} - \tilde{\mathbf{r}}_i \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{r}}_{i-1} \\ \dot{\boldsymbol{\omega}}_{i-1} \end{bmatrix} + \begin{bmatrix} \boldsymbol{\omega}_{i-1} \wedge \left[\boldsymbol{\omega}_{i-1} \wedge \left(\mathbf{r}_i - \mathbf{r}_{i-1}\right)\right] + \ddot{\mathbf{r}}_i^{i,i-1} + 2\boldsymbol{\omega}_{i-1} \wedge \dot{\mathbf{r}}_i^{i,i-1} \\ \boldsymbol{\alpha}_i^{i-1} + \boldsymbol{\omega}_{i-1} \wedge \boldsymbol{\omega}_i^{i-1} \end{bmatrix}$$

- Identifying terms:

$$\dot{\mathbf{V}}_i = \begin{bmatrix} \ddot{\mathbf{r}}_i \\ \dot{\boldsymbol{\omega}}_i \end{bmatrix}, \quad \dot{\mathbf{V}}_{i-1} = \begin{bmatrix} \ddot{\mathbf{r}}_{i-1} \\ \dot{\boldsymbol{\omega}}_{i-1} \end{bmatrix},$$

$$\mathbf{b}_i^v \ddot{\mathbf{z}}_i + \mathbf{d}_i^v = \begin{bmatrix} \boldsymbol{\omega}_{i-1} \wedge \left[\boldsymbol{\omega}_{i-1} \wedge \left(\mathbf{r}_i - \mathbf{r}_{i-1}\right)\right] + \ddot{\mathbf{r}}_i^{i,i-1} + 2\boldsymbol{\omega}_{i-1} \wedge \dot{\mathbf{r}}_i^{i,i-1} \\ \boldsymbol{\alpha}_i^{i-1} + \boldsymbol{\omega}_{i-1} \wedge \boldsymbol{\omega}_i^{i-1} \end{bmatrix}.$$

Dynamics of open-loop systems:

Recursive kinematic relations.

- Gathering linear and angular velocities:

$$\mathbf{V}_i = \mathbf{B}_i^v \mathbf{V}_{i-1} + \mathbf{b}_i^v \dot{\mathbf{z}}_i$$

- Similarly, accelerations can be expressed as:

$$\dot{\mathbf{V}}_i = \mathbf{B}_i^v \dot{\mathbf{V}}_{i-1} + \mathbf{b}_i^v \ddot{\mathbf{z}}_i + \mathbf{d}_i^v$$

- Recursive terms:

$$\mathbf{B}_i^v,\ \dot{\mathbf{B}}_i^v,\ \mathbf{b}_i^v,\ \dot{\mathbf{b}}_i^v.$$

- Regarding that:

$$\mathbf{d}_i^v = \dot{\mathbf{B}}_i^v \mathbf{V}_{i-1} + \dot{\mathbf{b}}_i^v \dot{\mathbf{z}}_i$$

Revolute joint:

$$\mathbf{b}_i^v = \begin{bmatrix} \tilde{\mathbf{w}}_j \left(\mathbf{r}_i - \mathbf{r}_j\right) \\ \mathbf{w}_j \end{bmatrix}$$

$$\dot{\mathbf{b}}_i^v = \begin{bmatrix} \dot{\tilde{\mathbf{w}}}_j \left(\mathbf{r}_i - \mathbf{r}_j\right) + \tilde{\mathbf{w}}_j \left(\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_j\right) \\ \dot{\mathbf{w}}_j \end{bmatrix}$$

$\boldsymbol{z}_i = [z_k]$ (angle).

Prismatic joint:

$$\mathbf{b}_i^v = \begin{bmatrix} \mathbf{u}_j \\ \mathbf{0} \end{bmatrix}$$

$$\dot{\mathbf{b}}_i^v = \begin{bmatrix} \dot{\mathbf{u}}_j \\ \mathbf{0} \end{bmatrix}$$

$\boldsymbol{z}_i = [z_k]$ (distance).

Cardan joint:

$$\mathbf{b}_i^v = \begin{bmatrix} \tilde{\mathbf{w}}_j \left(\mathbf{r}_i - \mathbf{r}_j\right) & \tilde{\mathbf{w}}_{j+1} \left(\mathbf{r}_i - \mathbf{r}_j\right) \\ \mathbf{w}_j & \mathbf{w}_{j+1} \end{bmatrix}$$

$$\dot{\mathbf{b}}_i^v = \begin{bmatrix} \dot{\mathbf{b}}_i^{v1} & \dot{\mathbf{b}}_i^{v2} \end{bmatrix}$$

$\boldsymbol{z}_i = [z_{k1}, z_{k2}]$ (angles).

Cylindrical joint:

$$\mathbf{b}_i^v = \begin{bmatrix} \mathbf{w}_j & \tilde{\mathbf{w}}_j \left(\mathbf{r}_i - \mathbf{r}_j\right) \\ \mathbf{0} & \mathbf{w}_j \end{bmatrix}$$

$$\dot{\mathbf{b}}_i^v = \begin{bmatrix} \dot{\mathbf{w}}_j & \dot{\tilde{\mathbf{w}}}_j \left(\mathbf{r}_i - \mathbf{r}_j\right) + \tilde{\mathbf{w}}_j \left(\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_j\right) \\ \mathbf{0} & \dot{\mathbf{w}}_j \end{bmatrix}$$

$\boldsymbol{z}_i = [z_{k1}, z_{k2}]$ (distance, angle).

Spherical joint:

$$\mathbf{b}_i^v = \begin{bmatrix} 2\left(\tilde{\mathbf{r}}_j - \tilde{\mathbf{r}}_i\right)\mathbf{E} \\ 2\mathbf{E} \end{bmatrix}$$

$$\dot{\mathbf{b}}_i^v = \begin{bmatrix} 2\left(\dot{\tilde{\mathbf{r}}}_j - \dot{\tilde{\mathbf{r}}}_i\right)\mathbf{E} + 2\left(\tilde{\mathbf{r}}_j - \tilde{\mathbf{r}}_i\right)\dot{\mathbf{E}} \\ 2\dot{\mathbf{E}} \end{bmatrix}$$

$\boldsymbol{z}_i = \overline{\boldsymbol{p}} = [\bar{e}_0, \bar{e}_1, \bar{e}_2, \bar{e}_3]$ (Euler parameters).

Floating joint:

$$\mathbf{b}_i^v = \begin{bmatrix} \mathbf{I}_3 & 2\left(\tilde{\mathbf{r}}_G^i - \tilde{\mathbf{r}}_i\right)\mathbf{E} \\ \mathbf{0} & 2\mathbf{E} \end{bmatrix}$$

$$\dot{\mathbf{b}}_1^v = \begin{bmatrix} \mathbf{0} & 2\left(\dot{\tilde{\mathbf{r}}}_G^i - \dot{\tilde{\mathbf{r}}}_i\right)\mathbf{E} + 2\left(\tilde{\mathbf{r}}_G^i - \tilde{\mathbf{r}}_i\right)\dot{\mathbf{E}} \\ \mathbf{0} & 2\dot{\mathbf{E}} \end{bmatrix}$$

$\boldsymbol{z}_i = [z_{k1}, z_{k2}, z_{k3}, \overline{\boldsymbol{p}}]$ (distances, Euler p.).

Planar joint:

$$\mathbf{b}_i^v = \begin{bmatrix} \mathbf{u}_j & \mathbf{v}_j & \tilde{\mathbf{w}}_j \left(\mathbf{r}_i - \mathbf{r}_G^i\right) \\ \mathbf{0} & \mathbf{0} & \mathbf{w}_j \end{bmatrix}$$

$$\dot{\mathbf{b}}_i^v = \begin{bmatrix} \dot{\mathbf{u}}_j & \dot{\mathbf{v}}_j & \dot{\tilde{\mathbf{w}}}_j \left(\mathbf{r}_i - \mathbf{r}_G^i\right) + \tilde{\mathbf{w}}_j \left(\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_G^i\right) \\ \mathbf{0} & \mathbf{0} & \dot{\mathbf{w}}_j \end{bmatrix}$$

$\boldsymbol{z}_i = [z_{k1}, z_{k2}, z_{k3}]$ (distance, distance, angle).

Recursive kinematic relations.

- Gathering linear and angular velocities:

$$\mathbf{V}_i = \mathbf{B}_i^v \mathbf{V}_{i-1} + \mathbf{b}_i^v \dot{\mathbf{z}}_i$$

- Similarly, accelerations can be expressed as:

$$\dot{\mathbf{V}}_i = \mathbf{B}_i^v \dot{\mathbf{V}}_{i-1} + \mathbf{b}_i^v \ddot{\mathbf{z}}_i + \mathbf{d}_i^v$$

- Recursive terms:

$$\mathbf{B}_i^v,\ \dot{\mathbf{B}}_i^v,\ \mathbf{b}_i^v,\ \dot{\mathbf{b}}_i^v.$$

- Regarding that:

$$\mathbf{d}_i^v = \dot{\mathbf{B}}_i^v \mathbf{V}_{i-1} + \dot{\mathbf{b}}_i^v \dot{\mathbf{z}}_i$$

- For the center of mass as reference point (RTdyn0):

$$\mathbf{Y}_i = \mathbf{B}_i^y \mathbf{Y}_{i-1} + \mathbf{b}_i^y \dot{\mathbf{z}}_i$$
$$\dot{\mathbf{Y}}_i = \mathbf{B}_i^y \dot{\mathbf{Y}}_{i-1} + \mathbf{b}_i^y \ddot{\mathbf{z}}_i + \mathbf{d}_i^y$$

- For the global origin of coordinates as reference point (RTdyn1):

$$\mathbf{Z}_i = \mathbf{Z}_{i-1} + \mathbf{b}_i^z \dot{\mathbf{z}}_i$$
$$\dot{\mathbf{Z}}_i = \dot{\mathbf{Z}}_{i-1} + \mathbf{b}_i^z \ddot{\mathbf{z}}_i + \mathbf{d}_i^z$$

- Regrouping terms for n_b bodies:

$$\left.\begin{aligned} \mathbf{V} &= \begin{bmatrix} \mathbf{V}_1^{\mathrm{T}} & \mathbf{V}_2^{\mathrm{T}} & \dots & \mathbf{V}_{n_b}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \\ \dot{\mathbf{z}} &= \begin{bmatrix} \dot{\mathbf{z}}_1^{\mathrm{T}} & \dot{\mathbf{z}}_2^{\mathrm{T}} & \dots & \dot{\mathbf{z}}_{n_b}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \end{aligned}\right\} \quad \mathbf{V} = \mathbf{R}^v \dot{\mathbf{z}}$$

Semi-recursive equations of motion.

- Applying the virtual power principle to a multibody system composed of n_b bodies :

$$\sum_{i=1}^{n_b} \begin{bmatrix} \dot{\mathbf{r}}_G^{i*} \\ \boldsymbol{\omega}_i^* \end{bmatrix} \left(\begin{bmatrix} m_i \mathbf{I}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_i^G \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{r}}_G^i \\ \dot{\boldsymbol{\omega}}_i \end{bmatrix} - \begin{bmatrix} \mathbf{f}_i \\ \mathbf{n}_i^G - \boldsymbol{\omega}_i \wedge \mathbf{J}_i^G \boldsymbol{\omega}_i \end{bmatrix} \right) = 0$$

- Extending the previous equation for any reference point:

$$\mathbf{Y}_i^* = \begin{bmatrix} \dot{\mathbf{r}}_G^{i*} \\ \boldsymbol{\omega}_i^* \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \tilde{\mathbf{r}}_i - \tilde{\mathbf{r}}_G^i \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{r}}_i \\ \boldsymbol{\omega}_i^* \end{bmatrix} = \mathbf{D}_i^v \mathbf{V}_i^*$$

$$\dot{\mathbf{Y}}_i = \begin{bmatrix} \ddot{\mathbf{r}}_G^i \\ \dot{\boldsymbol{\omega}}_i \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \tilde{\mathbf{r}}_i - \tilde{\mathbf{r}}_G^i \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{r}}_i \\ \dot{\boldsymbol{\omega}}_i \end{bmatrix} + \begin{bmatrix} \boldsymbol{\omega}_i \wedge \left(\boldsymbol{\omega}_i \wedge \left(\mathbf{r}_G^i - \mathbf{r}_i \right) \right) \\ \mathbf{0} \end{bmatrix} = \mathbf{D}_i^v \dot{\mathbf{V}}_i + \mathbf{e}_i^v$$

$$\sum_{i=1}^{n_b} \mathbf{V}_i^{*\mathrm{T}} \left[\mathbf{M}_i^v \dot{\mathbf{V}}_i - \mathbf{Q}_i^v \right] = 0$$

being: $\mathbf{M}_i^v = \left(\mathbf{D}_i^v\right)^{\mathrm{T}} \mathbf{M}_i \mathbf{D}_i^v$ and $\mathbf{Q}_i^v = \left(\mathbf{D}_i^v\right)^{\mathrm{T}} \left(\mathbf{Q}_i - \mathbf{M}_i \mathbf{e}_i^v\right)$

- Substituting $\mathbf{V}^* = \mathbf{R}^v \dot{\mathbf{z}}^*$ and $\dot{\mathbf{V}} = \mathbf{R}^v \ddot{\mathbf{z}} + \dot{\mathbf{R}}^v \dot{\mathbf{z}}$:

$$\dot{\mathbf{z}}^{*\mathrm{T}} \left[\left(\mathbf{R}^{v\mathrm{T}} \mathbf{M}^v \mathbf{R}^v \right) \ddot{\mathbf{z}} - \mathbf{R}^{v\mathrm{T}} \left(\mathbf{Q}^v - \mathbf{M}^v \dot{\mathbf{R}}^v \dot{\mathbf{z}} \right) \right] = 0$$

- Assuming that all the relative coordinate velocities are independent:

$$\left(\mathbf{R}^{v\mathrm{T}} \mathbf{M}^v \mathbf{R}^v \right) \ddot{\mathbf{z}} = \mathbf{R}^{v\mathrm{T}} \left(\mathbf{Q}^v - \mathbf{M}^v \dot{\mathbf{R}}^v \dot{\mathbf{z}} \right) \Rightarrow \mathbf{M}^d \ddot{\mathbf{z}} = \mathbf{Q}^d$$

Generic reference point.

- Mass matrix and force vector of each body:

$$\mathbf{M}_i^v = \begin{bmatrix} m_i\mathbf{I} & -m_i\left(\tilde{\mathbf{r}}_G^i - \tilde{\mathbf{r}}_i\right) \\ m_i\left(\tilde{\mathbf{r}}_G^i - \tilde{\mathbf{r}}_i\right) & \mathbf{J}_i^G - m_i\left(\tilde{\mathbf{r}}_G^i - \tilde{\mathbf{r}}_i\right)\left(\tilde{\mathbf{r}}_G^i - \tilde{\mathbf{r}}_i\right) \end{bmatrix}$$

$$\mathbf{Q}_i^v = \begin{bmatrix} \mathbf{f}_i - m_i\boldsymbol{\omega}_i \wedge \left(\boldsymbol{\omega}_i \wedge \left(\mathbf{r}_G^i - \mathbf{r}_i\right)\right) \\ \mathbf{n}_i^G - \boldsymbol{\omega}_i \wedge \mathbf{J}_i^G\boldsymbol{\omega}_i + \left(\mathbf{r}_G^i - \mathbf{r}_i\right) \wedge \left[\mathbf{f}_i - m_i\boldsymbol{\omega}_i \wedge \left(\boldsymbol{\omega}_i \wedge \left(\mathbf{r}_G^i - \mathbf{r}_i\right)\right)\right] \end{bmatrix}$$

- Assembly of matrix $\mathbf{R}^v$:

$$\mathbf{R}^v = \begin{bmatrix} \mathbf{b}_{1,1}^v & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{b}_{2,1}^v & \mathbf{b}_{2,2}^v & \dots & \mathbf{0} \\ \dots & \dots & \dots & \dots \\ \mathbf{b}_{n,1}^v & \mathbf{b}_{n,2}^v & \dots & \mathbf{b}_{n,n}^v \end{bmatrix}$$

$$\mathbf{b}_{i,i}^v = \mathbf{b}_i^v$$

$$\mathbf{b}_{i,j}^v = \mathbf{B}_i^v\mathbf{b}_{h,j}^v;\ i > j$$

$$\mathbf{b}_{i,j}^v = \mathbf{0};\ i < j$$

- Assembly of vector $\dot{\mathbf{R}}^v\dot{\mathbf{z}}$:

$$\dot{\mathbf{R}}^v\dot{\mathbf{z}} = \left[\left(\mathbf{d}_1^v\right)^{\mathrm{T}} \quad \left(\mathbf{d}_2^{v\Sigma}\right)^{\mathrm{T}} \quad \dots \quad \left(\mathbf{d}_n^{v\Sigma}\right)^{\mathrm{T}}\right]^{\mathrm{T}}$$

$$\mathbf{d}_i^{v\Sigma} = \mathbf{d}_i^v + \mathbf{B}_i^v\mathbf{d}_h^{v\Sigma}$$

- Assembly of generalized force vector $\mathbf{Q}^d$:

$$\mathbf{Q}_i^d = \mathbf{R}_i^{v\mathrm{T}}\left(\mathbf{Q}^v - \mathbf{M}^v\dot{\mathbf{R}}^v\dot{\mathbf{z}}\right) = \mathbf{b}_i^{v\mathrm{T}}\mathbf{Q}_i^{v\Sigma}$$

$$\mathbf{Q}_i^{v\Sigma} = \mathbf{Q}_i^v - \mathbf{M}_i^v\mathbf{d}_i^{v\Sigma} + \sum_{s=1}^{n_s^i}\mathbf{B}_s^{v\mathrm{T}}\mathbf{Q}_s^{v\Sigma}$$

- Assembly of mass matrix $\mathbf{M}^d$:

$$\mathbf{M}^d(i,j) = \mathbf{R}^{v\mathrm{T}}\mathbf{M}^v\mathbf{R}^v(i,j) = \mathbf{b}_i^{v\mathrm{T}}\mathbf{M}_i^{v\Sigma}\mathbf{b}_{i,j}^v; \qquad i > j,$$

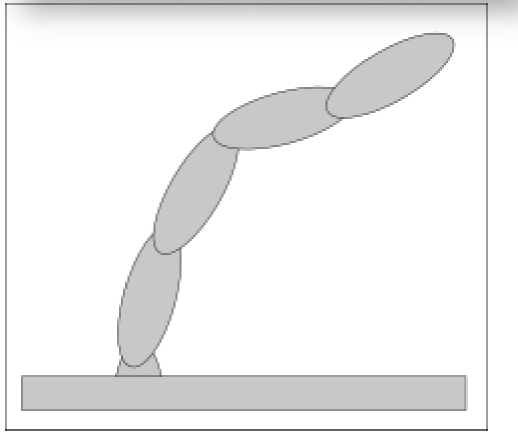
$$\mathbf{M}^d(i,i) = \mathbf{R}^{v\mathrm{T}}\mathbf{M}^v\mathbf{R}^v(i,i) = \mathbf{b}_i^{v\mathrm{T}}\mathbf{M}_i^{v\Sigma}\mathbf{b}_i^v,$$

$$\mathbf{M}^d(i,j) = \mathbf{R}^{v\mathrm{T}}\mathbf{M}^v\mathbf{R}^v(i,j) = \mathbf{b}_{j,i}^{v\mathrm{T}}\mathbf{M}_j^{v\Sigma}\mathbf{b}_j^v; \qquad i < j,$$

$$\mathbf{M}_i^{v\Sigma} = \mathbf{M}_i^v + \sum_{s=1}^{n_s^i}\mathbf{B}_s^{v\mathrm{T}}\mathbf{M}_s^{v\Sigma}\mathbf{B}_s^v$$

Fully-recursive equations of motion.

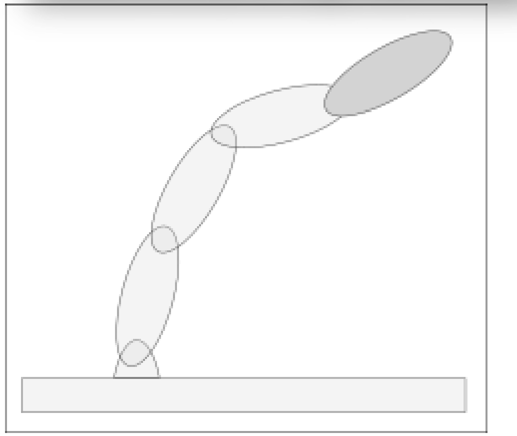
Forward kinematics



$$\mathbf{V}_i = \mathbf{B}_i^v \mathbf{V}_{i-1} + \mathbf{b}_i^v \dot{\mathbf{z}}_i$$

$$\dot{\mathbf{V}}_i = \mathbf{B}_i^v \dot{\mathbf{V}}_{i-1} + \mathbf{b}_i^v \ddot{\mathbf{z}}_i + \mathbf{d}_i^v$$

Backward dynamics



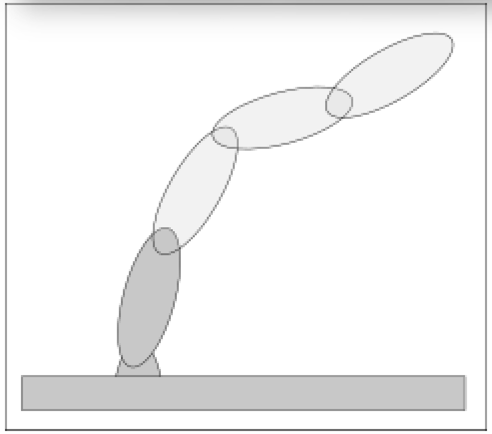
$$\hat{\mathbf{M}}_{i-1}^v = \mathbf{M}_{i-1}^v + \mathbf{B}_i^{v\mathrm{T}} \mathbf{K}_i \hat{\mathbf{M}}_i^v \mathbf{B}_i^v$$

$$\hat{\mathbf{Q}}_{i-1}^v = \mathbf{Q}_{i-1}^v + \mathbf{B}_i^{v\mathrm{T}} \left(\mathbf{K}_i \left(\hat{\mathbf{Q}}_i^v - \hat{\mathbf{M}}_i^v \mathbf{d}_i^v \right) + \mathbf{K}_i^{\mathbf{z}} \mathbf{Q}_i^{\mathbf{z}} \right)$$

$$\mathbf{K}_i^{\mathbf{z}} = -\hat{\mathbf{M}}_i^v \mathbf{b}_i^v \left[\mathbf{b}_i^{v\mathrm{T}} \hat{\mathbf{M}}_i^v \mathbf{b}_i^v \right]^{-1}$$

$$\mathbf{K}_i = \mathbf{I}_6 + \mathbf{K}_i^{\mathbf{z}} \mathbf{b}_i^{v\mathrm{T}}$$

Forward solution



$$\ddot{\mathbf{z}}_i = \left[\mathbf{b}_i^{v\mathrm{T}} \hat{\mathbf{M}}_i^v \mathbf{b}_i^v \right]^{-1} \left(\mathbf{b}_i^{v\mathrm{T}} \left[\hat{\mathbf{Q}}_i^v - \hat{\mathbf{M}}_i^v \left(\mathbf{B}_i^v \dot{\mathbf{V}}_{i-1} + \mathbf{d}_i^v \right) \right] + \mathbf{Q}_i^{\mathbf{z}} \right)$$

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3) **Dynamics of closed-loop systems**

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Constraints.

- **Euler parameter normalization constraint**.
- **Loop-closure constraints**:
  - Revolute joint
  - Prismatic joint
  - Cardan joint
  - Cylindrical joint
  - Spherical joint
  - Planar joint
- **User constraints**:
  - Driving constraints (rheonomous).
  - Geometric constraints.
  - Angle and distance definition, etc.

- **Constraint derivatives**:

$$\mathbf{\Phi}_{\hat{\mathbf{z}}} = \frac{\partial \mathbf{\Phi}}{\partial \mathbf{q}} \frac{\partial \mathbf{q}}{\partial \mathbf{z}} + \frac{\partial \mathbf{\Phi}}{\partial \mathbf{z}} = \mathbf{\Phi_q q_z} + \mathbf{\Phi_z}$$

  - **Explicit derivatives**: $\mathbf{\Phi_q}, \ \mathbf{\Phi_z}$
    - Explicit dependencies of constraint equations.
    - Constraint-dependent.
    - Efficiency: consider sparsity.
  - **Topological derivatives**: $\mathbf{q_z}$
    - Variation of natural coordinates with **z**.
    - Topology-dependent.
    - Efficiency: reuse and storage of topological derivatives.

Semi-recursive Matrix R formulation.

- **Kinematic velocity and acceleration problems**:

$$\begin{bmatrix} \boldsymbol{\Phi}_{\hat{\mathbf{z}}} \\ \mathbf{B} \end{bmatrix} \dot{\mathbf{z}} = \begin{bmatrix} \mathbf{b} \\ \dot{\mathbf{z}}^i \end{bmatrix}$$

$$\begin{bmatrix} \boldsymbol{\Phi}_{\hat{\mathbf{z}}} \\ \mathbf{B} \end{bmatrix} \ddot{\mathbf{z}} = \begin{bmatrix} \mathbf{c} \\ \ddot{\mathbf{z}}^i \end{bmatrix}$$

with $\mathbf{b} = -\boldsymbol{\Phi}_t, \quad \mathbf{c} = -\dot{\boldsymbol{\Phi}}_t - \dot{\boldsymbol{\Phi}}_{\hat{\mathbf{z}}}\dot{\mathbf{z}}.$

Solving these problems:

$$\dot{\mathbf{z}} = \mathbf{R}^{\boldsymbol{\Phi}}\dot{\mathbf{z}}^i + \mathbf{S}^{\boldsymbol{\Phi}}\mathbf{b}$$

$$\ddot{\mathbf{z}} = \mathbf{R}^{\boldsymbol{\Phi}}\ddot{\mathbf{z}}^i + \mathbf{S}^{\boldsymbol{\Phi}}\mathbf{c}$$

$$\dot{\mathbf{z}}^* = \mathbf{R}^{\boldsymbol{\Phi}}\dot{\mathbf{z}}^{*i}$$

in which:

$$\begin{bmatrix} \mathbf{S}^{\boldsymbol{\Phi}} & \mathbf{R}^{\boldsymbol{\Phi}} \end{bmatrix} \begin{bmatrix} \boldsymbol{\Phi}_{\hat{\mathbf{z}}} \\ \mathbf{B} \end{bmatrix} = \mathbf{I}_n$$

- **Semi-recursive Matrix R formulation**:
  - Set of independent coordinates ODE generated applying a second velocity projection ($\dot{\mathbf{z}} \rightarrow \dot{\mathbf{z}}^i$).
  - Classical formulation:

$$\left(\mathbf{R}^{\boldsymbol{\Phi}\mathrm{T}}\mathbf{M}^d\mathbf{R}^{\boldsymbol{\Phi}}\right)\ddot{\mathbf{z}}^i = \mathbf{R}^{\boldsymbol{\Phi}\mathrm{T}}\left(\mathbf{Q}^d - \mathbf{M}^d\mathbf{S}^{\boldsymbol{\Phi}}\mathbf{c}\right)$$

Semi-recursive Matrix R formulation.

- **Kinematic velocity and acceleration problems**:

$$\begin{bmatrix} \mathbf{\Phi}_{\hat{\mathbf{z}}} \\ \mathbf{B} \end{bmatrix} \dot{\mathbf{z}} = \begin{bmatrix} \mathbf{b} \\ \dot{\mathbf{z}}^i \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{\Phi}_{\hat{\mathbf{z}}} \\ \mathbf{B} \end{bmatrix} \ddot{\mathbf{z}} = \begin{bmatrix} \mathbf{c} \\ \ddot{\mathbf{z}}^i - \dot{\mathbf{B}}\dot{\mathbf{z}} \end{bmatrix}$$

with $\mathbf{b} = -\mathbf{\Phi}_t, \quad \mathbf{c} = -\dot{\mathbf{\Phi}}_t - \dot{\mathbf{\Phi}}_{\hat{\mathbf{z}}}\dot{\mathbf{z}}.$

Solving these problems:

$$\dot{\mathbf{z}} = \mathbf{R}^{\mathbf{\Phi}}\dot{\mathbf{z}}^i + \mathbf{S}^{\mathbf{\Phi}}\mathbf{b}$$

$$\ddot{\mathbf{z}} = \mathbf{R}^{\mathbf{\Phi}}\left(\ddot{\mathbf{z}}^i - \dot{\mathbf{B}}\dot{\mathbf{z}}\right) + \mathbf{S}^{\mathbf{\Phi}}\mathbf{c}$$

$$\dot{\mathbf{z}}^* = \mathbf{R}^{\mathbf{\Phi}}\dot{\mathbf{z}}^{*i}$$

in which:

$$\begin{bmatrix} \mathbf{S}^{\mathbf{\Phi}} & \mathbf{R}^{\mathbf{\Phi}} \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_{\hat{\mathbf{z}}} \\ \mathbf{B} \end{bmatrix} = \mathbf{I}_n$$

- **Semi-recursive Matrix R formulation**:
  - Set of independent coordinates ODE generated applying a second velocity projection ($\dot{\mathbf{z}} \rightarrow \dot{\mathbf{z}}^i$).
  - Classical formulation:

$$\left(\mathbf{R}^{\mathbf{\Phi}\mathrm{T}}\mathbf{M}^d\mathbf{R}^{\mathbf{\Phi}}\right)\ddot{\mathbf{z}}^i = \mathbf{R}^{\mathbf{\Phi}\mathrm{T}}\left(\mathbf{Q}^d - \mathbf{M}^d\mathbf{S}^{\mathbf{\Phi}}\mathbf{c}\right)$$

  - Novelty: non-constant B matrix, for DoF not included in the joint coordinates vector.

$$\mathbf{B} = \frac{\partial \mathbf{z}^i}{\partial \mathbf{z}}$$

  - Extended formulation:

$$\left(\mathbf{R}^{\mathbf{\Phi}\mathrm{T}}\mathbf{M}^d\mathbf{R}^{\mathbf{\Phi}}\right)\ddot{\mathbf{z}}^i = \mathbf{R}^{\mathbf{\Phi}\mathrm{T}}\left(\mathbf{Q}^d - \mathbf{M}^d\left(\mathbf{S}^{\mathbf{\Phi}}\mathbf{c} - \mathbf{R}^{\mathbf{\Phi}}\dot{\mathbf{B}}\dot{\mathbf{z}}\right)\right)$$

  - More compact:

$$\bar{\mathbf{M}}\ddot{\mathbf{z}}^i = \bar{\mathbf{Q}}$$

Semi-recursive ALI3-P formulation.

- **Augmented Lagrangian index-3**:

$$\mathbf{M}^d\ddot{\mathbf{z}}^* + \mathbf{\Phi}_{\hat{\mathbf{z}}}^{\mathrm{T}}\left(\boldsymbol{\lambda}^{*\{i+1\}} + \boldsymbol{\alpha}\mathbf{\Phi}\right) = \mathbf{Q}^d$$

$$\boldsymbol{\lambda}^{*\{i+1\}} = \boldsymbol{\lambda}^{*\{i\}} + \boldsymbol{\alpha}\mathbf{\Phi};\ i > 0$$

  - Once combined with a numerical integrator, it can be solved by means of a Newton-Raphson scheme in positions.
  - Supports redundant constraints.
  - Efficient and accurate.
  - Poor stability without energy preserving or energy decaying (dissipative) numerical integrators.

- **Velocity projection**:

$$\left(\bar{\mathbf{P}} + \varsigma\mathbf{\Phi}_{\hat{\mathbf{z}}}^{\mathrm{T}}\boldsymbol{\alpha}\mathbf{\Phi}_{\hat{\mathbf{z}}}\right)\dot{\mathbf{z}} = \bar{\mathbf{P}}\dot{\mathbf{z}}^* - \mathbf{\Phi}_{\hat{\mathbf{z}}}^{\mathrm{T}}\left(\boldsymbol{\sigma}^{\{i+1\}} + \varsigma\boldsymbol{\alpha}\mathbf{b}\right)$$

$$\boldsymbol{\sigma}^{\{i+1\}} = \boldsymbol{\sigma}^{\{i\}} + \varsigma\boldsymbol{\alpha}\dot{\mathbf{\Phi}}$$

  - Iterative or non-iterative.
  - Enforces the fulfilment of $\dot{\mathbf{\Phi}}$.

- **Acceleration projection:**

$$\left(\bar{\mathbf{P}} + \varsigma\mathbf{\Phi}_{\hat{\mathbf{z}}}^{\mathrm{T}}\boldsymbol{\alpha}\mathbf{\Phi}_{\hat{\mathbf{z}}}\right)\ddot{\mathbf{z}} = \bar{\mathbf{P}}\ddot{\mathbf{z}}^* - \mathbf{\Phi}_{\hat{\mathbf{z}}}^{\mathrm{T}}\left(\boldsymbol{\kappa}^{\{i+1\}} + \varsigma\boldsymbol{\alpha}\mathbf{c}\right)$$

$$\boldsymbol{\kappa}^{\{i+1\}} = \boldsymbol{\kappa}^{\{i\}} + \varsigma\boldsymbol{\alpha}\ddot{\mathbf{\Phi}}$$

  - Iterative or non-iterative.
  - Enforces the fulfilment of $\ddot{\mathbf{\Phi}}$.

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1) Introduction and motivation
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3) Dynamics of closed-loop systems
4) **Sensitivity analysis of unconstrained open-loop systems**
5) Sensitivity analysis of closed-loop systems
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Notation.

- Objective function:

$$\psi = \int_{t_0}^{t_F} \mathbf{g}\left(\mathbf{z}, \dot{\mathbf{z}}, \ddot{\mathbf{z}}, \mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}, \boldsymbol{\rho}\right) \mathrm{d}t$$

- Gradient:

$$\psi' = \nabla\psi^T = \int_{t_0}^{t_F} \left(\mathbf{g_z}\mathbf{z}' + \mathbf{g_{\dot{z}}}\dot{\mathbf{z}}' + \mathbf{g_{\ddot{z}}}\ddot{\mathbf{z}}' + \mathbf{g_q}\mathbf{q}' + \mathbf{g_{\dot{q}}}\dot{\mathbf{q}}' + \mathbf{g_{\ddot{q}}}\ddot{\mathbf{q}}' + \mathbf{g}_{\boldsymbol{\rho}}\right) \mathrm{d}t$$

- Considering implicit dependencies:

$$\mathbf{q}' = \mathbf{q_z}\mathbf{z}' + \mathbf{q}_{\boldsymbol{\rho}}$$
$$\dot{\mathbf{q}}' = \dot{\mathbf{q}}_\mathbf{z}\mathbf{z}' + \mathbf{q_z}\dot{\mathbf{z}}' + \dot{\mathbf{q}}_{\boldsymbol{\rho}}$$
$$\ddot{\mathbf{q}}' = \ddot{\mathbf{q}}_\mathbf{z}\mathbf{z}' + 2\dot{\mathbf{q}}_\mathbf{z}\dot{\mathbf{z}}' + \mathbf{q_z}\ddot{\mathbf{z}}' + \ddot{\mathbf{q}}_{\boldsymbol{\rho}}$$

- The gradient can be expressed as:

$$\psi' = \int_{t_0}^{t_F} \left(\mathbf{g}_{\hat{\mathbf{z}}}\mathbf{z}' + \mathbf{g}_{\hat{\dot{\mathbf{z}}}}\dot{\mathbf{z}}' + \mathbf{g}_{\hat{\ddot{\mathbf{z}}}}\ddot{\mathbf{z}}' + \mathbf{g}_{\hat{\boldsymbol{\rho}}}\right) \mathrm{d}t$$

wherein:

$$\mathbf{g}_{\hat{\mathbf{z}}} = \mathbf{g_z} + \mathbf{g_q}\mathbf{q_z} + \mathbf{g_{\dot{q}}}\dot{\mathbf{q}}_\mathbf{z} + \mathbf{g_{\ddot{q}}}\ddot{\mathbf{q}}_\mathbf{z}$$
$$\mathbf{g}_{\hat{\dot{\mathbf{z}}}} = \mathbf{g_{\dot{z}}} + \mathbf{g_{\dot{q}}}\mathbf{q_z} + 2\mathbf{g_{\ddot{q}}}\dot{\mathbf{q}}_\mathbf{z}$$
$$\mathbf{g}_{\hat{\ddot{\mathbf{z}}}} = \mathbf{g_{\ddot{z}}} + \mathbf{g_{\ddot{q}}}\mathbf{q_z}$$
$$\mathbf{g}_{\hat{\boldsymbol{\rho}}} = \mathbf{g_q}\mathbf{q}_{\boldsymbol{\rho}} + \mathbf{g_{\dot{q}}}\dot{\mathbf{q}}_{\boldsymbol{\rho}} + \mathbf{g_{\ddot{q}}}\ddot{\mathbf{q}}_{\boldsymbol{\rho}} + \mathbf{g}_{\boldsymbol{\rho}}$$

- Notation:
  - It gathers implicit dependencies on natural coordinates.
  - It is compact.

Sensitivity analysis of unconstrained open-loop systems:

Semi-recursive forward sensitivity.

- Taking derivatives on the semi-recursive open-loop EoM:

$$\left(\mathbf{M}^d_{\hat{\mathbf{z}}}\ddot{\mathbf{z}} + \mathbf{K}\right)\mathbf{z}' + \mathbf{C}\dot{\mathbf{z}}' + \mathbf{M}^d\ddot{\mathbf{z}}' = \mathbf{Q}^d_{\hat{\boldsymbol{\rho}}} - \mathbf{M}^d_{\hat{\boldsymbol{\rho}}}\ddot{\mathbf{z}}$$

with:

$$\mathbf{K} = -\mathbf{Q}^d_{\hat{\mathbf{z}}}$$

$$\mathbf{C} = -\mathbf{Q}^d_{\hat{\dot{\mathbf{z}}}}$$

- The systems of sensitivity equations:
  - Need a numerical integrator to be solved.
  - Require the exact derivatives of the mass matrix and the generalized forces vector.
  - Involve *n×p* variables.

Dynamics

$$\mathbf{M}^d\ddot{\mathbf{z}} = \mathbf{Q}^d$$

Derivative assessment

$$\mathbf{M}^d_{\hat{\mathbf{z}}},\ \mathbf{M}^d_{\hat{\boldsymbol{\rho}}},\ \mathbf{K},\ \mathbf{C},\ \mathbf{Q}^d_{\hat{\boldsymbol{\rho}}}.$$

Numerical integrator

$$\dot{\mathbf{z}}'_{n+1} = \frac{\gamma}{\beta h}\mathbf{z}'_{n+1} + \hat{\dot{\mathbf{z}}}'_n$$

$$\ddot{\mathbf{z}}'_{n+1} = \frac{1}{\beta h^2}\mathbf{z}'_{n+1} + \hat{\ddot{\mathbf{z}}}'_n$$

Sensitivity solution

$$\left(\mathbf{M}^d + \gamma h\mathbf{C}^d + \beta h^2\left(\mathbf{M}^d_{\hat{\mathbf{z}}}\ddot{\mathbf{z}} + \mathbf{K}^d\right)\right)\mathbf{z}' = \beta h^2\left(\mathbf{Q}^d_{\hat{\boldsymbol{\rho}}} - \mathbf{M}^d_{\hat{\boldsymbol{\rho}}}\ddot{\mathbf{z}} - \mathbf{C}^d\hat{\dot{\mathbf{z}}}' - \mathbf{M}^d\hat{\ddot{\mathbf{z}}}'\right)$$

Gradient integration

$$\psi'(t_i) = \psi'(t_{i-1}) + \frac{h}{2}\left(\dot{\psi}'(t_{i-1}) + \dot{\psi}'(t_i)\right)$$

$$\dot{\psi}'(t_i) = \left\{\mathbf{g}_{\hat{\mathbf{z}}}\mathbf{z}' + \mathbf{g}_{\hat{\dot{\mathbf{z}}}}\dot{\mathbf{z}}' + \mathbf{g}_{\hat{\ddot{\mathbf{z}}}}\ddot{\mathbf{z}}' + \mathbf{g}_{\hat{\boldsymbol{\rho}}}\right\}_{t_i}$$

Mass matrix derivatives.

Body mass matrix → **Accumulated mass matrix** → **Assembly**

$$\left(\mathbf{M}_i^v\right)_{\hat{\mathbf{x}}} = \left(\left(\mathbf{D}_i^v\right)^{\mathrm{T}} \mathbf{M}_i \mathbf{D}_i^v\right)_{\hat{\mathbf{x}}}$$

$$\left(\mathbf{M}_i^\Sigma\right)_{\hat{\mathbf{x}}} = \left(\mathbf{M}_i^v + \sum_{j=1}^{n_s^i} \left(\mathbf{B}_s^{v\,\mathrm{T}} \mathbf{M}_s^{v\Sigma} \mathbf{B}_s^v\right)\right)_{\hat{\mathbf{x}}}$$

$$\left(\mathbf{M}^d(i,j)\right)_{\hat{\mathbf{x}}} = \left(\mathbf{b}_i^{v\mathrm{T}} \mathbf{M}_i^\Sigma \mathbf{b}_{i,j}^v\right)_{\hat{\mathbf{x}}}$$

- RTdyn0:

$$\mathbf{D}_i^y = \mathbf{I},\ \left(\mathbf{D}_i^y\right)_{\hat{\mathbf{z}}} = \mathbf{0},\ \left(\mathbf{D}_i^y\right)_{\hat{\boldsymbol{\rho}}} = \mathbf{0}.$$

- RTdyn1:

$$\mathbf{B}_i^z = \mathbf{I},\ \left(\mathbf{B}_i^z\right)_{\hat{\mathbf{z}}} = \mathbf{0},\ \left(\mathbf{B}_i^z\right)_{\hat{\boldsymbol{\rho}}} = \mathbf{0}.$$

Forces derivatives.

Body forces vector → **Accumulated forces vector** → **Assembly**

- External forces
- Inertial forces

$$\left(\mathbf{Q}_i^{v\Sigma}\right)_{\hat{\mathbf{x}}} = \left(\mathbf{Q}_i^{v} - \mathbf{M}_i^{v}\mathbf{d}_i^{v\Sigma} + \sum_{s=1}^{n_s^i} \mathbf{B}_s^{v\mathrm{T}}\mathbf{Q}_s^{v\Sigma}\right)_{\hat{\mathbf{x}}}$$

$$\left(\mathbf{Q}_i^{d}\right)_{\hat{\mathbf{x}}} = \left(\mathbf{b}_i^{v\mathrm{T}}\mathbf{Q}_i^{v\Sigma}\right)_{\hat{\mathbf{x}}}$$

$$\left(\mathbf{Q}_i^{v}\right)_{\hat{\mathbf{x}}} = \left(\sum_{j=1}^{n_f^i} \mathbf{Q}_{i,j}^{v\;(e)} + \mathbf{Q}_i^{v\,(I)}\right)_{\hat{\mathbf{x}}}$$

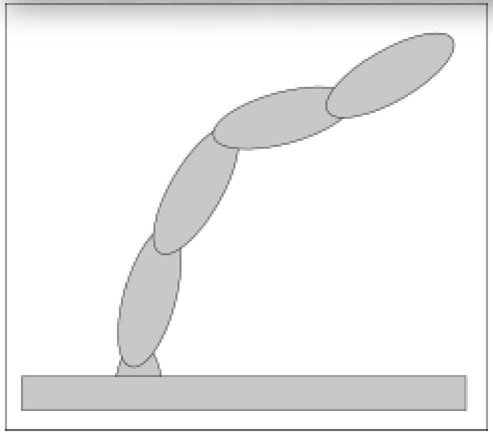
$$\left(\mathbf{Q}_{i,j}^{v\;(e)}\right)_{\hat{\mathbf{x}}} = \begin{bmatrix} \mathbf{f}_j \\ \mathbf{n}_j^G + \left(\tilde{\mathbf{r}}_G^i - \tilde{\mathbf{r}}_i\right)\mathbf{f}_j \end{bmatrix}_{\hat{\mathbf{x}}}$$

$$\left(\mathbf{Q}_i^{v\,(I)}\right)_{\hat{\mathbf{x}}} = \begin{bmatrix} -m_i\tilde{\boldsymbol{\omega}}_i\left(\tilde{\boldsymbol{\omega}}_i\left(\mathbf{r}_G^i - \mathbf{r}_i\right)\right) \\ -\tilde{\boldsymbol{\omega}}_i\mathbf{J}_i^G\boldsymbol{\omega}_i - \left(\tilde{\mathbf{r}}_G^i - \tilde{\mathbf{r}}_i\right)\left(m_i\tilde{\boldsymbol{\omega}}_i\left(\tilde{\boldsymbol{\omega}}_i\left(\mathbf{r}_G^i - \mathbf{r}_i\right)\right)\right) \end{bmatrix}_{\hat{\mathbf{x}}}$$

Sensitivity analysis of unconstrained open-loop systems:

Fully-recursive forward sensitivity.

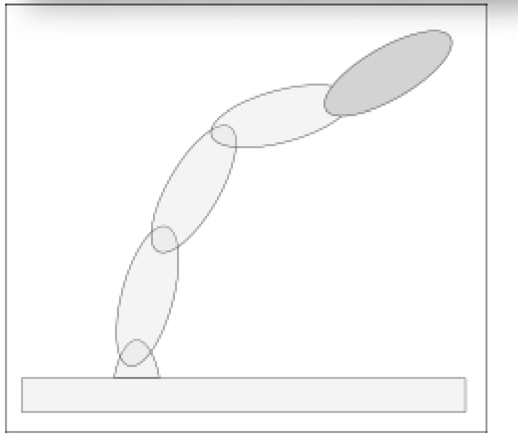
Kinematic derivatives



$$\mathbf{V}_i, \dot{\mathbf{V}}_i, \mathbf{B}_i^v, \dot{\mathbf{B}}_i^v, \mathbf{b}_i^v, \dot{\mathbf{b}}_i^v, \mathbf{d}_i^v, \mathbf{A}_i$$

- Derivatives with respect to positions, velocities and parameters.

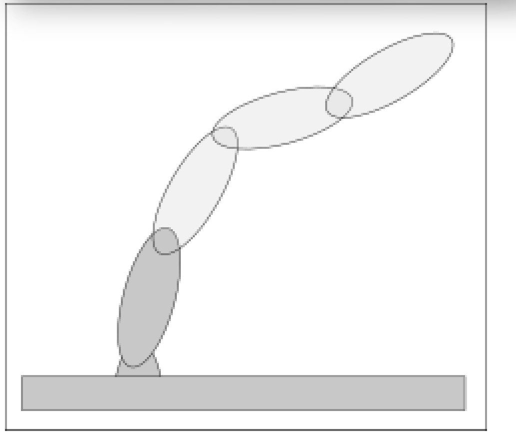
Dynamic derivatives



$$\hat{\mathbf{M}}_i^v, \hat{\mathbf{Q}}_i^v, \mathbf{K}_i^{\mathbf{z}}, \mathbf{K}_i.$$

- Concatenation of products.
- Study each magnitude separately for efficiency.

Forward sensitivities



$$\ddot{\mathbf{z}}_i' = \left(\ddot{\mathbf{z}}_i\right)_{\hat{\mathbf{z}}}\mathbf{z}' + \left(\ddot{\mathbf{z}}_i\right)_{\hat{\dot{\mathbf{z}}}}\dot{\mathbf{z}}' + \left(\ddot{\mathbf{z}}_i\right)_{\hat{\ddot{\mathbf{z}}}}\ddot{\mathbf{z}}' + \left(\ddot{\mathbf{z}}_i\right)_{\hat{\boldsymbol{\rho}}}$$

- Recursive evaluation.
- Position and velocity sensitivities numerically integrated.

Derivatives: recursive terms.

Derivatives of recursive terms

Joint dependent

$\mathbf{A}_i^{i-1},\ \mathbf{b}_i^v,\ \dot{\mathbf{b}}_i^v.$

- w.r.t. $\mathbf{z}$
  - $\left(\mathbf{A}_i^{i-1}\right)_{\mathbf{z}}$
  - $\left(\mathbf{b}_i^v\right)_{\hat{\mathbf{z}}}$
  - $\left(\dot{\mathbf{b}}_i^v\right)_{\hat{\mathbf{z}}}$
- w.r.t. $\dot{\mathbf{z}}$
  - $\left(\dot{\mathbf{b}}_i^v\right)_{\hat{\dot{\mathbf{z}}}}$
- w.r.t. $\boldsymbol{\rho}$
  - $\left(\mathbf{A}_i^{i-1}\right)_{\rho}$
  - $\left(\mathbf{b}_i^v\right)_{\hat{\rho}}$
  - $\left(\dot{\mathbf{b}}_i^v\right)_{\hat{\rho}}$

Joint independent

$\mathbf{d}_i^v,\ \mathbf{B}_i^v,\ \dot{\mathbf{B}}_i^v.$

- w.r.t. $\mathbf{z}$
  - $\left(\mathbf{B}_i^v\right)_{\hat{\mathbf{z}}}$
  - $\left(\dot{\mathbf{B}}_i^v\right)_{\hat{\mathbf{z}}}$
  - $\left(\mathbf{d}_i^v\right)_{\hat{\mathbf{z}}}$
- w.r.t. $\dot{\mathbf{z}}$
  - $\left(\dot{\mathbf{B}}_i^v\right)_{\hat{\dot{\mathbf{z}}}}$
  - $\left(\mathbf{d}_i^v\right)_{\hat{\dot{\mathbf{z}}}}$
- w.r.t. $\boldsymbol{\rho}$
  - $\left(\mathbf{B}_i^v\right)_{\hat{\rho}}$
  - $\left(\dot{\mathbf{B}}_i^v\right)_{\hat{\rho}}$
  - $\left(\mathbf{d}_i^v\right)_{\hat{\rho}}$

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Semi-recursive Matrix R. Forward sensitivity.

- **Sensitivity of kinematic problems**:

$$\begin{bmatrix} \mathbf{\Phi}_{\hat{\mathbf{z}}} \\ \mathbf{B} \end{bmatrix} \mathbf{z}' = \begin{bmatrix} -\mathbf{\Phi}_{\hat{\rho}} \\ \mathbf{z}^{i\prime} \end{bmatrix} \Rightarrow \mathbf{z}' = \mathbf{R}^{\mathbf{\Phi}} \mathbf{z}^{i\prime} - \mathbf{S}^{\mathbf{\Phi}} \mathbf{\Phi}_{\hat{\rho}}$$

$$\begin{bmatrix} \mathbf{\Phi}_{\hat{\mathbf{z}}} \\ \mathbf{B} \end{bmatrix} \dot{\mathbf{z}}' = \begin{bmatrix} -\mathbf{b}^{\rho} \\ \dot{\mathbf{z}}^{i\prime} - \bar{\mathbf{b}}^{\rho} \end{bmatrix} \Rightarrow \dot{\mathbf{z}}' = \mathbf{R}^{\mathbf{\Phi}} \left(\dot{\mathbf{z}}^{i\prime} - \bar{\mathbf{b}}^{\rho} \right) - \mathbf{S}^{\mathbf{\Phi}} \mathbf{b}^{\rho}$$

$$\begin{bmatrix} \mathbf{\Phi}_{\hat{\mathbf{z}}} \\ \mathbf{B} \end{bmatrix} \ddot{\mathbf{z}}' = \begin{bmatrix} -\mathbf{c}^{\rho} \\ \ddot{\mathbf{z}}^{i\prime} - \bar{\mathbf{c}}^{\rho} \end{bmatrix} \Rightarrow \ddot{\mathbf{z}}' = \mathbf{R}^{\mathbf{\Phi}} \left(\ddot{\mathbf{z}}^{i\prime} - \bar{\mathbf{c}}^{\rho} \right) - \mathbf{S}^{\mathbf{\Phi}} \mathbf{c}^{\rho}$$

Expanding these expressions:

$$\mathbf{z}' = \mathbf{R}^{\mathbf{\Phi}} \mathbf{z}^{i\prime} - \mathbf{S}^{\mathbf{\Phi}} \mathbf{\Phi}_{\hat{\rho}}$$

$$\dot{\mathbf{z}}' = \mathbf{R}^{\mathbf{\Phi}} \left(\dot{\mathbf{z}}^{i\prime} - \dot{\mathbf{B}} \mathbf{z}' + \mathbf{B}_{\hat{\rho}} \dot{\mathbf{z}} \right) - \mathbf{S}^{\mathbf{\Phi}} \left(\dot{\mathbf{\Phi}}_{\hat{\mathbf{z}}} \mathbf{z}' + \dot{\mathbf{\Phi}}_{\hat{\rho}} \right)$$

$$\begin{aligned} \ddot{\mathbf{z}}' = \mathbf{R}^{\mathbf{\Phi}} & \left(\ddot{\mathbf{z}}^{i\prime} - 2\dot{\mathbf{B}} \dot{\mathbf{z}}' + \ddot{\mathbf{B}} \mathbf{z}' + \mathbf{B}_{\hat{\rho}} \ddot{\mathbf{z}} + \dot{\mathbf{B}}_{\hat{\rho}} \dot{\mathbf{z}} \right) \\ & - \mathbf{S}^{\mathbf{\Phi}} \left(2\dot{\mathbf{\Phi}}_{\hat{\mathbf{z}}} \dot{\mathbf{z}}' + \ddot{\mathbf{\Phi}}_{\hat{\mathbf{z}}} \mathbf{z}' + \ddot{\mathbf{\Phi}}_{\hat{\rho}} \right) \end{aligned}$$

- **Forward sensitivity of semi-recursive Matrix R**:
  - The TLM takes the form:

$$\bar{\mathbf{M}} \ddot{\mathbf{z}}^{i\prime} + \bar{\mathbf{C}} \dot{\mathbf{z}}^{i\prime} + \left(\bar{\mathbf{K}} + \bar{\mathbf{M}}_{\mathbf{z}^i} \ddot{\mathbf{z}}^i \right) \mathbf{z}^{i\prime} = \bar{\mathbf{Q}}_{\check{\rho}} - \bar{\mathbf{M}}_{\check{\rho}} \ddot{\mathbf{z}}^i$$

with:

$$\bar{\mathbf{Q}}_{\check{\rho}} - \bar{\mathbf{M}}_{\check{\rho}} \ddot{\mathbf{z}}^i = \bar{\mathbf{Q}}_{\hat{\rho}} - \bar{\mathbf{M}}_{\hat{\rho}} \ddot{\mathbf{z}}^i + \left(\bar{\mathbf{Q}}_{\hat{\mathbf{z}}} + \bar{\mathbf{Q}}_{\hat{\dot{\mathbf{z}}}} \dot{\mathbf{z}}_{\mathbf{z}} - \bar{\mathbf{M}}_{\hat{\mathbf{z}}} \ddot{\mathbf{z}}^i \right) \mathbf{z}_{\rho} + \bar{\mathbf{Q}}_{\hat{\dot{\mathbf{z}}}} \dot{\mathbf{z}}_{\rho}$$

$$\bar{\mathbf{K}} = -\bar{\mathbf{Q}}_{\mathbf{z}^i} = -\left(\bar{\mathbf{Q}}_{\hat{\mathbf{z}}} + \bar{\mathbf{Q}}_{\hat{\dot{\mathbf{z}}}} \dot{\mathbf{z}}_{\mathbf{z}} \right) \mathbf{z}_{\mathbf{z}^i}$$

$$\bar{\mathbf{C}} = -\bar{\mathbf{Q}}_{\dot{\mathbf{z}}^i} = -\bar{\mathbf{Q}}_{\hat{\dot{\mathbf{z}}}} \dot{\mathbf{z}}_{\dot{\mathbf{z}}^i}$$

$$\bar{\mathbf{M}}_{\mathbf{z}^i} \ddot{\mathbf{z}}^i = \left(\bar{\mathbf{M}}_{\hat{\mathbf{z}}} \ddot{\mathbf{z}}^i \right) \mathbf{z}_{\mathbf{z}^i}$$

In which a new notation involving joint coordinate implicit dependencies has been used.

Semi-recursive Matrix R. Adjoint sensitivity.

- First, the EoM are transformed into a **first order explicit system**:

$$\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{M}} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{z}}^i \\ \dot{\mathbf{v}} \end{bmatrix} = \begin{bmatrix} \mathbf{v} \\ \bar{\mathbf{Q}} \end{bmatrix}$$

$$\hat{\mathbf{M}}(\mathbf{y},\boldsymbol{\rho})\,\dot{\mathbf{y}} = \hat{\mathbf{Q}}(t,\mathbf{y},\boldsymbol{\rho})$$

thus:

$$\dot{\mathbf{y}} = \hat{\mathbf{M}}^{-1}(\mathbf{y},\boldsymbol{\rho})\,\hat{\mathbf{Q}}(t,\mathbf{y},\boldsymbol{\rho}) = \mathbf{f}(t,\mathbf{y},\boldsymbol{\rho})$$

- The adjoint sensitivity begins with the definition of the following **Lagrangian**:

$$\mathcal{L}(\boldsymbol{\rho}) = \psi - \int_{t_0}^{t_F} \boldsymbol{\mu}^{\mathrm{T}}\left(\dot{\mathbf{y}} - \mathbf{f}(t,\mathbf{y},\boldsymbol{\rho})\right)\mathrm{d}t$$

with $\boldsymbol{\mu}$ a set of new adjoint variables.

- After taking derivatives, integrating by parts in time and nullifying terms multiplying the sensitivities of the states:

$$\dot{\boldsymbol{\mu}} = -\mathbf{f}_{\check{\mathbf{y}}}{}^{\mathrm{T}}\left(\boldsymbol{\mu} + \mathbf{g}_{\check{\dot{\mathbf{y}}}}{}^{\mathrm{T}}\right) - \mathbf{g}_{\check{\mathbf{y}}}{}^{\mathrm{T}}$$

$$\boldsymbol{\mu}^{t_F} = \mathbf{0}$$

$$\mathbf{f}_{\check{\mathbf{y}}} = \hat{\mathbf{M}}^{-1}\left(\hat{\mathbf{Q}}_{\check{\mathbf{y}}} - \hat{\mathbf{M}}_{\check{\mathbf{y}}}\mathbf{f}\right) = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\bar{\mathbf{M}}^{-1}\left(\bar{\mathbf{K}} + \bar{\mathbf{M}}_{\mathbf{z}^i}\dot{\mathbf{v}}\right) & -\bar{\mathbf{M}}^{-1}\bar{\mathbf{C}} \end{bmatrix}$$

$$\mathbf{g}_{\check{\mathbf{y}}} = \begin{bmatrix} \mathbf{g}_{\check{\mathbf{z}}^i} & \mathbf{g}_{\check{\dot{\mathbf{z}}}^i} \end{bmatrix}$$

$$\mathbf{g}_{\check{\dot{\mathbf{y}}}} = \begin{bmatrix} \mathbf{0} & \mathbf{g}_{\check{\ddot{\mathbf{z}}}^i} \end{bmatrix}$$

- Finally, the **objective function gradient** can be determined with:

$$\psi'^{\mathrm{T}} = -\left[\mathbf{y}_{\check{\boldsymbol{\rho}}}^{\mathrm{T}}\boldsymbol{\mu}\right]_{t_0} + \int_{t_0}^{t_F}\left(\mathbf{f}_{\check{\boldsymbol{\rho}}}^{\mathrm{T}}\left(\boldsymbol{\mu} + \mathbf{g}_{\check{\dot{\mathbf{y}}}}{}^{\mathrm{T}}\right) + \mathbf{g}_{\check{\boldsymbol{\rho}}}^{\mathrm{T}}\right)\mathrm{d}t$$

$$\mathbf{f}_{\check{\boldsymbol{\rho}}} = \hat{\mathbf{M}}^{-1}\left(\hat{\mathbf{Q}}_{\check{\boldsymbol{\rho}}} - \hat{\mathbf{M}}_{\check{\boldsymbol{\rho}}}\mathbf{f}\right) = \begin{bmatrix} \mathbf{0} \\ \bar{\mathbf{M}}^{-1}\left(\bar{\mathbf{Q}}_{\check{\boldsymbol{\rho}}} - \bar{\mathbf{M}}_{\check{\boldsymbol{\rho}}}\ddot{\mathbf{z}}^i\right) \end{bmatrix}$$

Semi-recursive ALI3-P. Forward sensitivity.

ALI3 sensitivity

$$\left[\mathbf{M}^d \ddot{\mathbf{z}}^{*\prime\{i\}} + \mathbf{C}\dot{\mathbf{z}}^{*\prime\{i\}} + \bar{\mathbf{K}}\mathbf{z}^{\prime\{i\}} + \boldsymbol{\Phi}_{\hat{\mathbf{z}}}^{\mathrm{T}} \boldsymbol{\lambda}^{*\prime\{i\}}\right] = \bar{\mathbf{Q}}^{\rho}$$

$$\boldsymbol{\lambda}^{*\prime\{i\}} = \boldsymbol{\lambda}^{*\prime\{i-1\}} + \alpha\boldsymbol{\Phi}'$$

$$\bar{\mathbf{K}} = \mathbf{M}^d_{\hat{\mathbf{z}}}\ddot{\mathbf{z}}^* + \boldsymbol{\Phi}^{\mathrm{T}}_{\hat{\mathbf{z}}\hat{\mathbf{z}}}\left(\boldsymbol{\lambda}^* + \alpha\boldsymbol{\Phi}\right) + \boldsymbol{\Phi}^{\mathrm{T}}_{\hat{\mathbf{z}}}\alpha\boldsymbol{\Phi}_{\hat{\mathbf{z}}} + \mathbf{K}$$

$$\bar{\mathbf{Q}}^{\rho} = \mathbf{Q}^d_{\hat{\rho}} - \mathbf{M}^d_{\hat{\rho}}\ddot{\mathbf{z}}^* - \boldsymbol{\Phi}^{\mathrm{T}}_{\hat{\mathbf{z}}\hat{\rho}}\left(\boldsymbol{\lambda}^* + \alpha\boldsymbol{\Phi}\right) - \boldsymbol{\Phi}^{\mathrm{T}}_{\hat{\mathbf{z}}}\alpha\boldsymbol{\Phi}_{\hat{\rho}}$$

$$\boldsymbol{\Phi}' = \boldsymbol{\Phi}_{\hat{\mathbf{z}}}\mathbf{z}' + \boldsymbol{\Phi}_{\hat{\rho}}$$

Sensitivity of velocity projection

$$\left(\bar{\mathbf{P}} + \varsigma\boldsymbol{\Phi}^{\mathrm{T}}_{\hat{\mathbf{z}}}\alpha\boldsymbol{\Phi}_{\hat{\mathbf{z}}}\right)\dot{\mathbf{z}}'^{\{i\}} = \bar{\mathbf{P}}\dot{\mathbf{z}}^{*\prime} + \bar{\mathbf{P}}'\left(\dot{\mathbf{z}}^* - \dot{\mathbf{z}}\right) - \boldsymbol{\Phi}^{\mathrm{T}}_{\hat{\mathbf{z}}\hat{\mathbf{z}}}\left(\boldsymbol{\sigma} + \varsigma\alpha\dot{\boldsymbol{\Phi}}\right)\mathbf{z}' - \boldsymbol{\Phi}^{\mathrm{T}}_{\hat{\mathbf{z}}\hat{\rho}}\left(\boldsymbol{\sigma} + \varsigma\alpha\dot{\boldsymbol{\Phi}}\right) - \boldsymbol{\Phi}^{\mathrm{T}}_{\hat{\mathbf{z}}}\left(\boldsymbol{\sigma}'^{\{i\}} + \varsigma\alpha\mathbf{b}^{\rho}\right)$$

$$\boldsymbol{\sigma}'^{\{i\}} = \boldsymbol{\sigma}'^{\{i-1\}} + \varsigma\alpha\dot{\boldsymbol{\Phi}}';\ i > 1$$

Sensitivity of acceleration projection

$$\left(\bar{\mathbf{P}} + \varsigma\boldsymbol{\Phi}^{\mathrm{T}}_{\hat{\mathbf{z}}}\alpha\boldsymbol{\Phi}_{\hat{\mathbf{z}}}\right)\ddot{\mathbf{z}}'^{\{i\}} = \bar{\mathbf{P}}\ddot{\mathbf{z}}^{*\prime} + \bar{\mathbf{P}}'\left(\ddot{\mathbf{z}}^* - \ddot{\mathbf{z}}\right) - \boldsymbol{\Phi}^{\mathrm{T}}_{\hat{\mathbf{z}}\hat{\mathbf{z}}}\left(\boldsymbol{\kappa} + \varsigma\alpha\ddot{\boldsymbol{\Phi}}\right)\mathbf{z}' - \boldsymbol{\Phi}^{\mathrm{T}}_{\hat{\mathbf{z}}\hat{\rho}}\left(\boldsymbol{\kappa} + \varsigma\alpha\ddot{\boldsymbol{\Phi}}\right) - \boldsymbol{\Phi}^{\mathrm{T}}_{\hat{\mathbf{z}}}\left(\boldsymbol{\kappa}'^{\{i\}} + \varsigma\alpha\mathbf{c}^{\rho}\right)$$

$$\boldsymbol{\kappa}'^{\{i\}} = \boldsymbol{\kappa}'^{\{i-1\}} + \varsigma\alpha\ddot{\boldsymbol{\Phi}}';\ i > 1$$

Semi-recursive ALI3-P. Adjoint sensitivity.

Adjoint Variable Method (AVM)

→ **Continuous AVM (differentiate-then-discretize)**

- Continuous EoM.
- Differential adjoint equations.
- General expressions.

→ **Discrete AVM (discretize-then-differentiate)**

- Discretized EoM.
- Algebraic adjoint equations.
- Particular expressions.

Semi-recursive ALI3-P. Continuous adjoint sensitivity.

Continuous adjoint Lagrangian

Classical index-3 formulation

$$\int_{t_0}^{t_F} \boldsymbol{\mu}_2^{\mathrm{T}} \left(\mathbf{M}\dot{\mathbf{v}}^* + \boldsymbol{\Phi}_{\hat{\mathbf{z}}}^{\mathrm{T}} \left(\boldsymbol{\lambda}^* + \alpha\boldsymbol{\Phi}\right) - \mathbf{Q}\right) \mathrm{d}t \qquad \int_{t_0}^{t_F} \boldsymbol{\mu}_{\boldsymbol{\Phi}}^{\mathrm{T}} \boldsymbol{\Phi} \mathrm{d}t$$

Velocity projection

$$\int_{t_0}^{t_F} \boldsymbol{\mu}_{\dot{\boldsymbol{\Phi}}}^{\mathrm{T}} \left(\left[\bar{\mathbf{P}} + \varsigma\boldsymbol{\Phi}_{\hat{\mathbf{z}}}^{\mathrm{T}}\alpha\boldsymbol{\Phi}_{\hat{\mathbf{z}}}\right]\dot{\mathbf{z}} - \bar{\mathbf{P}}\mathbf{v}^* + \boldsymbol{\Phi}_{\hat{\mathbf{z}}}^{\mathrm{T}}\varsigma\alpha\boldsymbol{\Phi}_t\right) \mathrm{d}t$$

Acceleration projection

$$\int_{t_0}^{t_F} \boldsymbol{\mu}_{\ddot{\boldsymbol{\Phi}}}^{\mathrm{T}} \left(\left[\bar{\mathbf{P}} + \varsigma\boldsymbol{\Phi}_{\hat{\mathbf{z}}}^{\mathrm{T}}\alpha\boldsymbol{\Phi}_{\hat{\mathbf{z}}}\right]\ddot{\mathbf{z}} - \bar{\mathbf{P}}\dot{\mathbf{v}}^* + \boldsymbol{\Phi}_{\hat{\mathbf{z}}}\varsigma\alpha\left(\dot{\boldsymbol{\Phi}}_{\hat{\mathbf{z}}}\dot{\mathbf{z}} + \dot{\boldsymbol{\Phi}}_t\right)\right) \mathrm{d}t$$

Variable change

$$\int_{t_0}^{t_F} \boldsymbol{\mu}_1^{\mathrm{T}} \left(\dot{\mathbf{z}}^* - \mathbf{v}^*\right) \mathrm{d}t$$

Final condition

$$\boldsymbol{\eta}^{\mathrm{T}}\dot{\boldsymbol{\Phi}}\left(t_F, \mathbf{q}_F, \mathbf{v}_F^*, \boldsymbol{\rho}\right)$$

- The adjoint equations take the form:

$$\dot{\boldsymbol{\mu}}_1 - \bar{\mathbf{K}}^{\mathrm{T}}\boldsymbol{\mu}_2 - \boldsymbol{\Phi}_{\hat{\mathbf{z}}}^{\mathrm{T}}\boldsymbol{\mu}_{\boldsymbol{\Phi}} - \mathbf{A}^{\mathrm{T}}\boldsymbol{\mu}_{\dot{\boldsymbol{\Phi}}} - \mathbf{B}^{\mathrm{T}}\boldsymbol{\mu}_{\ddot{\boldsymbol{\Phi}}} = -\mathbf{g}_{\hat{\mathbf{z}}}^{\mathrm{T}},$$

$$\mathbf{M}^{\mathrm{T}}\dot{\boldsymbol{\mu}}_2 - \bar{\mathbf{P}}^{\mathrm{T}}\dot{\boldsymbol{\mu}}_{\dot{\boldsymbol{\Phi}}} - \bar{\mathbf{C}}^{\mathrm{T}}\boldsymbol{\mu}_2 + \boldsymbol{\mu}_1 + \mathbf{E}^{\mathrm{T}}\boldsymbol{\mu}_{\dot{\boldsymbol{\Phi}}} + \mathbf{F}^{\mathrm{T}}\boldsymbol{\mu}_{\ddot{\boldsymbol{\Phi}}} = \mathbf{0},$$

$$\left(\bar{\mathbf{P}} + \boldsymbol{\Phi}_{\hat{\mathbf{z}}}^{\mathrm{T}}\varsigma\alpha\boldsymbol{\Phi}_{\hat{\mathbf{z}}}\right)^{\mathrm{T}}\boldsymbol{\mu}_{\dot{\boldsymbol{\Phi}}} + 2\varsigma\left(\boldsymbol{\Phi}_{\hat{\mathbf{z}}}^{\mathrm{T}}\alpha\dot{\boldsymbol{\Phi}}_{\hat{\mathbf{z}}}\right)^{\mathrm{T}}\boldsymbol{\mu}_{\ddot{\boldsymbol{\Phi}}} = \mathbf{g}_{\dot{\hat{\mathbf{z}}}}^{\mathrm{T}},$$

$$\left(\bar{\mathbf{P}} + \boldsymbol{\Phi}_{\hat{\mathbf{z}}}^{\mathrm{T}}\varsigma\alpha\boldsymbol{\Phi}_{\hat{\mathbf{z}}}\right)^{\mathrm{T}}\boldsymbol{\mu}_{\ddot{\boldsymbol{\Phi}}} = \mathbf{g}_{\ddot{\hat{\mathbf{z}}}}^{\mathrm{T}},$$

$$\boldsymbol{\Phi}_{\hat{\mathbf{z}}}\boldsymbol{\mu}_2 = \mathbf{g}_{\boldsymbol{\lambda}}^{\mathrm{T}}.$$

$$\mathbf{A} = \boldsymbol{\Phi}_{\hat{\mathbf{z}}\hat{\mathbf{z}}}^{\mathrm{T}}\varsigma\alpha\dot{\boldsymbol{\Phi}} - \bar{\mathbf{P}}_{\hat{\mathbf{z}}}\left(\mathbf{v}^* - \dot{\mathbf{z}}\right) + \boldsymbol{\Phi}_{\hat{\mathbf{z}}}^{\mathrm{T}}\varsigma\alpha\dot{\boldsymbol{\Phi}}_{\hat{\mathbf{z}}},$$

$$\mathbf{B} = \boldsymbol{\Phi}_{\hat{\mathbf{z}}\hat{\mathbf{z}}}^{\mathrm{T}}\varsigma\alpha\ddot{\boldsymbol{\Phi}} - \bar{\mathbf{P}}_{\hat{\mathbf{z}}}\left(\dot{\mathbf{v}}^* - \ddot{\mathbf{z}}\right) + \boldsymbol{\Phi}_{\hat{\mathbf{z}}}^{\mathrm{T}}\varsigma\alpha\ddot{\boldsymbol{\Phi}}_{\hat{\mathbf{z}}},$$

$$\bar{\mathbf{C}} = \mathbf{C} - \dot{\mathbf{M}},$$

$$\mathbf{E} = \bar{\mathbf{P}} + \bar{\mathbf{P}}_{\mathbf{v}^*}\left(\mathbf{v}^* - \dot{\mathbf{z}}\right),$$

$$\mathbf{F} = -\dot{\bar{\mathbf{P}}} + \bar{\mathbf{P}}_{\mathbf{v}^*}\left(\dot{\mathbf{v}}^* - \ddot{\mathbf{z}}\right),$$

$$\bar{\mathbf{K}} = \mathbf{K} + \mathbf{M}_{\hat{\mathbf{z}}}\dot{\mathbf{v}}^* + \boldsymbol{\Phi}_{\hat{\mathbf{z}}}^{\mathrm{T}}\alpha\boldsymbol{\Phi}_{\hat{\mathbf{z}}} + \boldsymbol{\Phi}_{\hat{\mathbf{z}}\hat{\mathbf{z}}}^{\mathrm{T}}\left(\boldsymbol{\lambda}^* + \alpha\boldsymbol{\Phi}\right).$$

Semi-recursive ALI3-P. Discrete adjoint sensitivity.

Discrete adjoint Lagrangian

Classical index-3 formulation

$$\boldsymbol{\mu}^{\mathrm{T}}\left(\mathbf{M}\ddot{\mathbf{z}}^{*}+\boldsymbol{\Phi}_{\hat{\mathbf{z}}}^{\mathrm{T}}\left(\boldsymbol{\lambda}^{*}+\alpha\boldsymbol{\Phi}\right)-\mathbf{Q}\right) \qquad \boldsymbol{\mu}_{\boldsymbol{\Phi}}^{\mathrm{T}}\boldsymbol{\Phi}$$

Velocity projection

$$\boldsymbol{\mu}_{\dot{\boldsymbol{\Phi}}}^{\mathrm{T}}\left(\left[\bar{\mathbf{P}}+\varsigma\boldsymbol{\Phi}_{\hat{\mathbf{z}}}^{\mathrm{T}}\alpha\boldsymbol{\Phi}_{\hat{\mathbf{z}}}\right]\dot{\mathbf{z}}-\bar{\mathbf{P}}\dot{\mathbf{z}}^{*}+\boldsymbol{\Phi}_{\hat{\mathbf{z}}}^{\mathrm{T}}\varsigma\alpha\boldsymbol{\Phi}_{t}\right)$$

Acceleration projection

$$\boldsymbol{\mu}_{\ddot{\boldsymbol{\Phi}}}^{\mathrm{T}}\left(\left[\bar{\mathbf{P}}+\varsigma\boldsymbol{\Phi}_{\hat{\mathbf{z}}}^{\mathrm{T}}\alpha\boldsymbol{\Phi}_{\hat{\mathbf{z}}}\right]\ddot{\mathbf{z}}-\bar{\mathbf{P}}\ddot{\mathbf{z}}^{*}+\boldsymbol{\Phi}_{\hat{\mathbf{z}}}\varsigma\alpha\left(\dot{\boldsymbol{\Phi}}_{\hat{\mathbf{z}}}\dot{\mathbf{z}}+\dot{\boldsymbol{\Phi}}_{t}\right)\right)$$

- The discrete systems of adjoint equations for the Newmark numerical integrator are:

$$\left\{\mathbf{T}^{\mathrm{T}}\boldsymbol{\mu}+\boldsymbol{\Phi}_{\hat{\mathbf{z}}}^{\mathrm{T}}\boldsymbol{\mu}_{\boldsymbol{\Phi}}-\mathbf{A}_{\dot{\boldsymbol{\Phi}}}^{\mathrm{T}}\boldsymbol{\mu}_{\dot{\boldsymbol{\Phi}}}-\mathbf{A}_{\ddot{\boldsymbol{\Phi}}}^{\mathrm{T}}\boldsymbol{\mu}_{\ddot{\boldsymbol{\Phi}}}\right\}_{\{i\}}=\mathbf{g}_{\hat{\mathbf{z}}\{i\}}^{\mathrm{T}}+\left\{\frac{\gamma}{\beta h}\mathbf{G}^{\mathrm{T}}+\frac{1}{\beta h^{2}}\mathbf{H}^{\mathrm{T}}\right\}_{\{i+1\}}$$

$$\left\{\left(\bar{\mathbf{P}}+\varsigma\boldsymbol{\Phi}_{\hat{\mathbf{z}}}^{\mathrm{T}}\alpha\boldsymbol{\Phi}_{\hat{\mathbf{z}}}\right)^{\mathrm{T}}\boldsymbol{\mu}_{\dot{\boldsymbol{\Phi}}}+\left(\boldsymbol{\Phi}_{\hat{\mathbf{z}}}^{\mathrm{T}}\varsigma\alpha\left(2\dot{\boldsymbol{\Phi}}_{\hat{\mathbf{z}}}\right)\right)^{\mathrm{T}}\boldsymbol{\mu}_{\ddot{\boldsymbol{\Phi}}}\right\}_{\{i\}}=\mathbf{g}_{\hat{\dot{\mathbf{z}}}\{i\}}^{\mathrm{T}}-\left\{\left(1-\frac{\gamma}{\beta}\right)\mathbf{G}^{\mathrm{T}}-\frac{1}{\beta h}\mathbf{H}^{\mathrm{T}}\right\}_{\{i+1\}}$$

$$\left\{\left(\bar{\mathbf{P}}+\varsigma\boldsymbol{\Phi}_{\hat{\mathbf{z}}}^{\mathrm{T}}\alpha\boldsymbol{\Phi}_{\hat{\mathbf{z}}}\right)^{\mathrm{T}}\boldsymbol{\mu}_{\ddot{\boldsymbol{\Phi}}}\right\}_{\{i\}}=\mathbf{g}_{\hat{\ddot{\mathbf{z}}}\{i\}}^{\mathrm{T}}-\left\{\left(1-\frac{\gamma}{2\beta}\right)h\mathbf{G}^{\mathrm{T}}+\left(1-\frac{1}{2\beta}\right)\mathbf{H}^{\mathrm{T}}\right\}_{\{i+1\}}$$

$$\left\{\boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{\Phi}_{\hat{\mathbf{z}}}^{\mathrm{T}}\right\}_{\{i\}}=\mathbf{g}_{\boldsymbol{\lambda}^{*}\{i\}}$$

$$\mathbf{T}=\frac{1}{\beta h^{2}}\mathbf{M}^{d}+\frac{\gamma}{\beta h}\bar{\mathbf{C}}+\bar{\mathbf{K}}$$

$$\mathbf{G}=\boldsymbol{\mu}^{\mathrm{T}}\mathbf{C}-\boldsymbol{\mu}_{\dot{\boldsymbol{\Phi}}}^{\mathrm{T}}\left(\bar{\mathbf{P}}+\bar{\mathbf{P}}_{\hat{\dot{\mathbf{z}}}^{*}}\left(\dot{\mathbf{z}}^{*}-\dot{\mathbf{z}}\right)\right)-\boldsymbol{\mu}_{\ddot{\boldsymbol{\Phi}}}^{\mathrm{T}}\bar{\mathbf{P}}_{\hat{\dot{\mathbf{z}}}^{*}}\left(\ddot{\mathbf{z}}^{*}-\ddot{\mathbf{z}}\right)$$

$$\mathbf{H}=\boldsymbol{\mu}^{\mathrm{T}}\mathbf{M}^{d}-\boldsymbol{\mu}_{\ddot{\boldsymbol{\Phi}}}^{\mathrm{T}}\bar{\mathbf{P}}$$

$$\mathbf{A}_{\dot{\boldsymbol{\Phi}}}=\bar{\mathbf{P}}\frac{\gamma}{\beta h}+\left(\bar{\mathbf{P}}_{\hat{\mathbf{z}}}+\frac{\gamma}{\beta h}\bar{\mathbf{P}}_{\hat{\dot{\mathbf{z}}}^{*}}\right)\left(\dot{\mathbf{z}}^{*}-\dot{\mathbf{z}}\right)-\boldsymbol{\Phi}_{\hat{\mathbf{z}}\hat{\mathbf{z}}}^{\mathrm{T}}\varsigma\alpha\dot{\boldsymbol{\Phi}}-\boldsymbol{\Phi}_{\hat{\mathbf{z}}}^{\mathrm{T}}\varsigma\alpha\dot{\boldsymbol{\Phi}}_{\hat{\mathbf{z}}}$$

$$\mathbf{A}_{\ddot{\boldsymbol{\Phi}}}=\bar{\mathbf{P}}\frac{1}{\beta h^{2}}+\left(\bar{\mathbf{P}}_{\hat{\mathbf{z}}}+\frac{\gamma}{\beta h}\bar{\mathbf{P}}_{\hat{\dot{\mathbf{z}}}^{*}}\right)\left(\ddot{\mathbf{z}}^{*}-\ddot{\mathbf{z}}\right)-\boldsymbol{\Phi}_{\hat{\mathbf{z}}\hat{\mathbf{z}}}^{\mathrm{T}}\varsigma\alpha\ddot{\boldsymbol{\Phi}}-\boldsymbol{\Phi}_{\hat{\mathbf{z}}}^{\mathrm{T}}\varsigma\alpha\ddot{\boldsymbol{\Phi}}_{\hat{\mathbf{z}}}$$

- **DAVM:**
  - No high order derivatives.
  - Straightforward initialization.
  - Particular adjoint equations.

Outline

1) Introduction and motivation
2) Dynamics of open-loop systems
3) Dynamics of closed-loop systems
4) Sensitivity analysis of unconstrained open-loop systems
5) Sensitivity analysis of closed-loop systems
6) MBSLIM implementation
7) Numerical experiments
8) Conclusions

MBSLIM implementation.

- **Objectives**:
  - **Unique** model definition.
  - Relative coordinate models **automatically generated** from the user information.
  - **Coexistence** of natural and relative coordinate models.
- **Topological kinematics implementation**:
  - **Automatic detection** of joints from information given in terms of points and vectors.
  - Detection and cutting of **closed loops**.
  - Solution of **constrained kinematic problems** in joint coordinates.
- **Topological dynamic implementation**:
  - Fully-recursive unconstrained dynamics.
  - Semi-recursive unconstrained dynamics.
  - Semi-recursive Matrix R and ALI3-P constrained dynamics.
- **Topological sensitivity analysis implementation**:
  - Forward fully-recursive and semi-recursive unconstrained sensitivity.
  - Forward and adjoint semi-recursive Matrix R sensitivity.
  - Forward, continuous adjoint and discrete adjoint semi-recursive ALI3-P sensitivity.

Outline

1) Introduction and motivation
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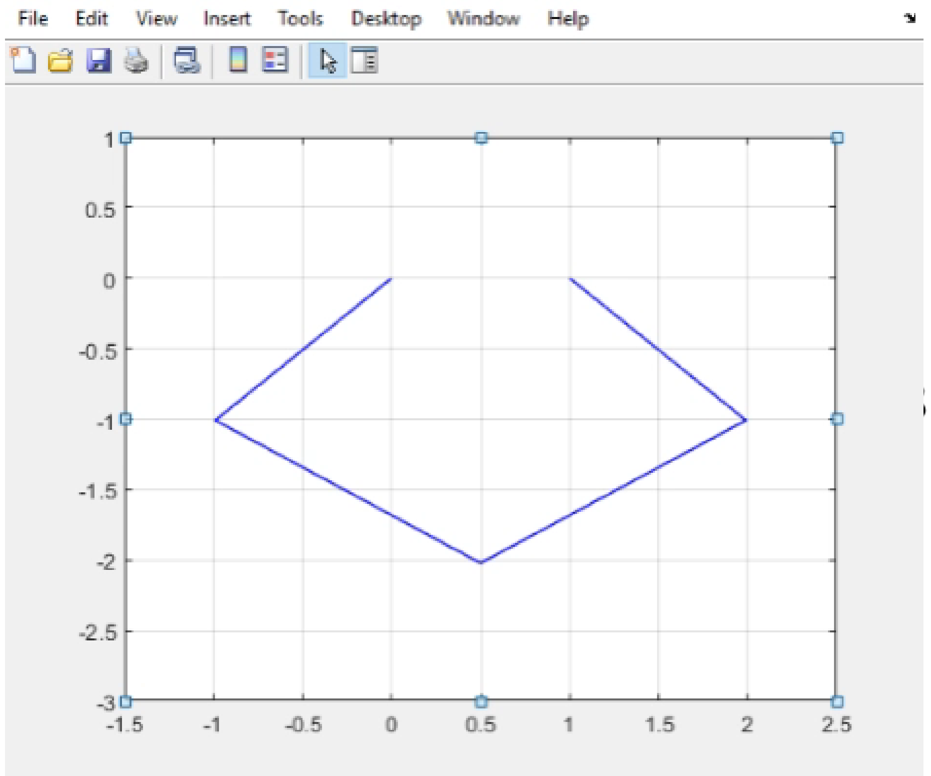
7) **Numerical experiments**
8) Conclusions

Five-bar: model.



- **Experiment:**
  - Dynamic maneuver during 5 seconds using semi-recursive Matrix R and ALI3-P formulations.

- **4 movable bodies.**
- **2 degrees of freedom.**
- **Forces:**
  - 2 spring-damper forces.
  - Gravitational forces.
- **Natural coordinates model:**
  - 23 variables.
  - 24 constraints.
  - Modelled with 5 points, 8 vectors and 2 angles.
- **Relative coordinates model:**
  - 4 variables.
  - 6 redundant constraints (loop closure).
  - Modelled with 4 revolute joints.

Five-bar: sensitivity results.

- **Objective function**:

$$\boldsymbol{\psi} = \begin{bmatrix} \psi^1 \\ \psi^2 \\ \psi^3 \end{bmatrix}$$

with:

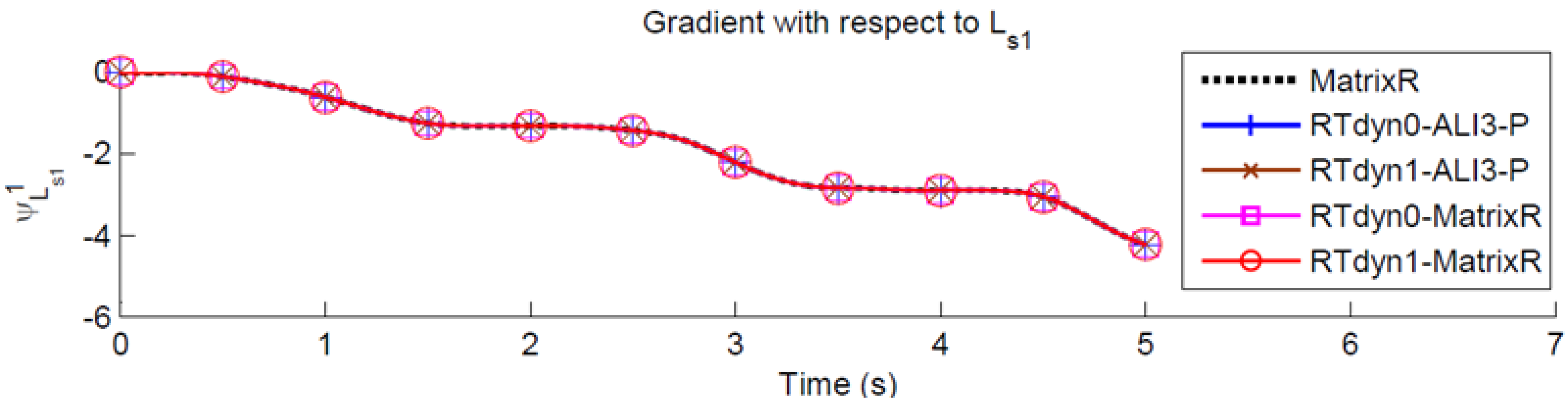
$$\psi^1 = \int_{t_0}^{t_F} (\mathbf{r}_2 - \mathbf{r}_{20})^{\mathrm{T}} (\mathbf{r}_2 - \mathbf{r}_{20})\, \mathrm{d}t$$

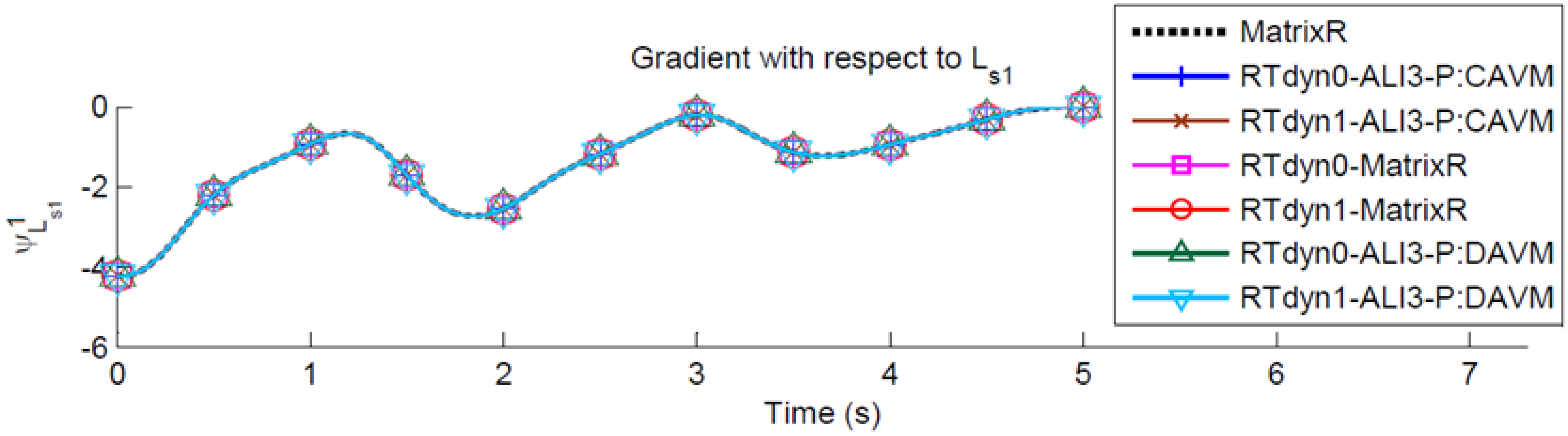
$$\psi^2 = \int_{t_0}^{t_F} \dot{\mathbf{r}}_2^{\mathrm{T}} \dot{\mathbf{r}}_2 \mathrm{d}t$$

$$\psi^3 = \int_{t_0}^{t_F} \ddot{\mathbf{r}}_2^{\mathrm{T}} \ddot{\mathbf{r}}_2 \mathrm{d}t$$

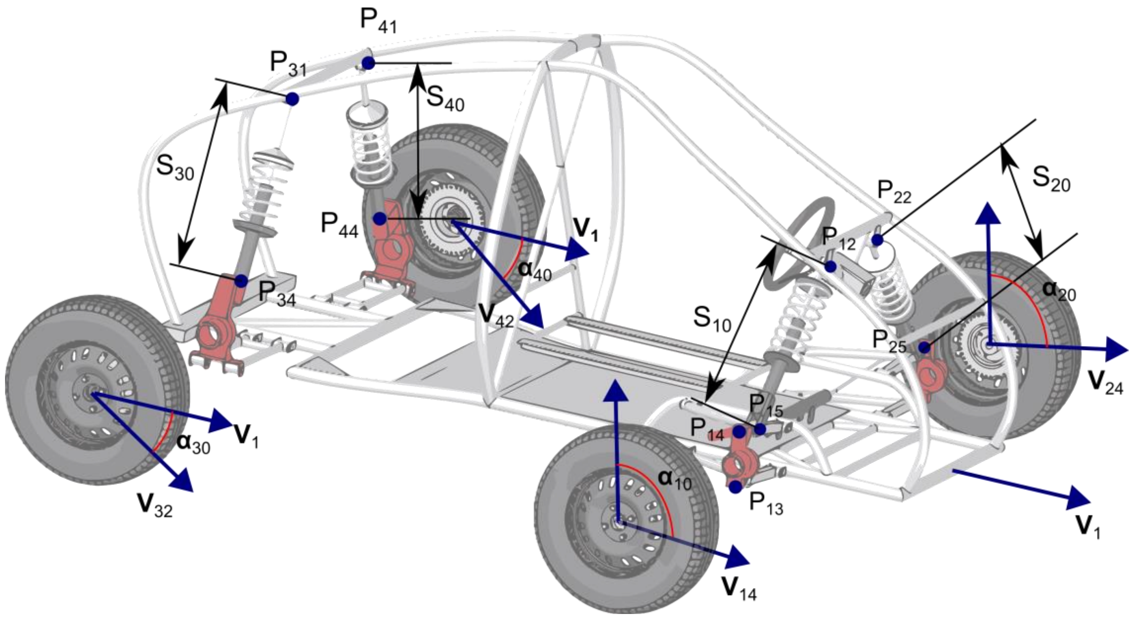
- **Parameters**:

$$\boldsymbol{\rho} = \begin{bmatrix} L_{s1} & L_{s2} & m_{A1} & r^G & L_{A1} \end{bmatrix}^{\mathrm{T}}$$





Numerical experiments:
Buggy vehicle: model.

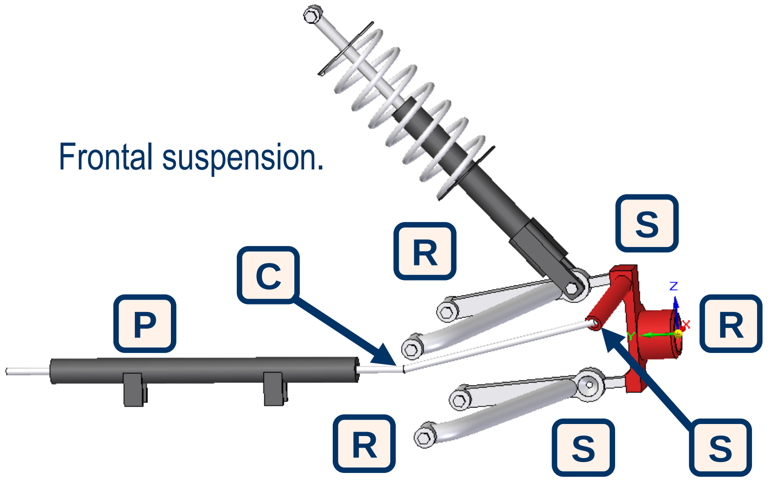


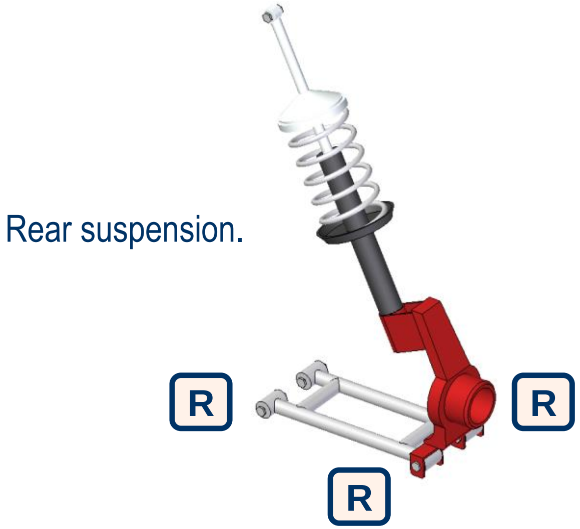
- **Experiments:**
  1. Step descent maneuver during 4.5 seconds.
  2. Double lane change during 12 seconds.

- **18 bodies.**
- **14 degrees of freedom.**
- **Additional constraints:**
  - Steering guide.
  - Definition of spin angle on each wheel.
  - Definition of distance on each suspension.
  - Alignment of 3 points on each rear suspension.
- **Forces:**
  - Weight of each body.
  - Spring-damper forces on front and rear suspension.
  - Tire forces.
- **Natural coordinates model:**
  - 180 variables.
  - 178 constraints.
  - Modelled with 32 points and 25 vectors.

Buggy vehicle: joint coordinates model.

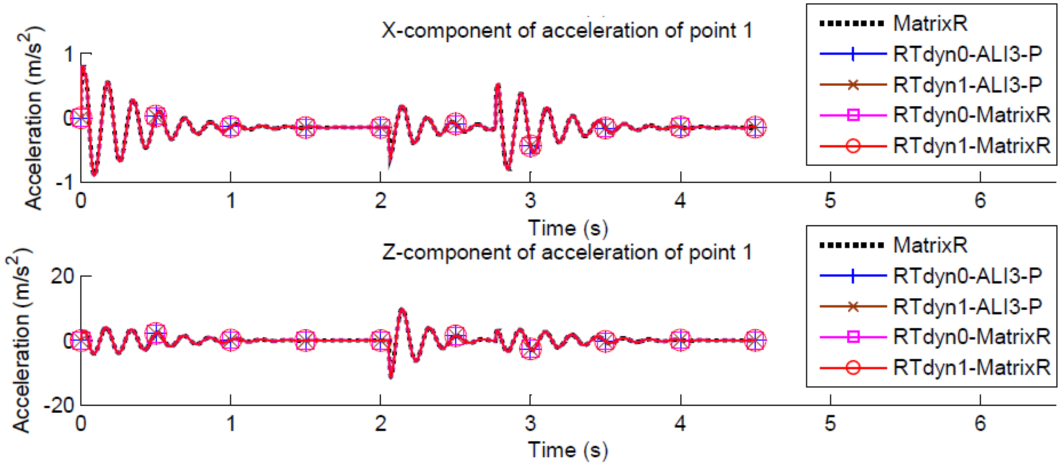
- **1 floating joint between ground and chassis.**
- **Each front suspension:**
  - 2 revolute with chassis
  - 4 spherical joints (2 eliminated).
- **Each rear suspension:**
  - 3 revolute joints and a constraint of alignment of 3 points.
- **Steering system:**
  - 1 prismatic joint
  - 2 cardan joints
  - 2 spherical joints (removed during the opening of closed loops).
- **1 floating + 12 revolute + 2 spherical + 1 prismatic + 2 Cardan.**
- **Joint coordinates model:**
  - 36 relative coordinates: 32 from joints and 4 from added variables.
  - 26 constraint equations.





Numerical experiments:
Buggy vehicle: step descent maneuver.

- Dynamics

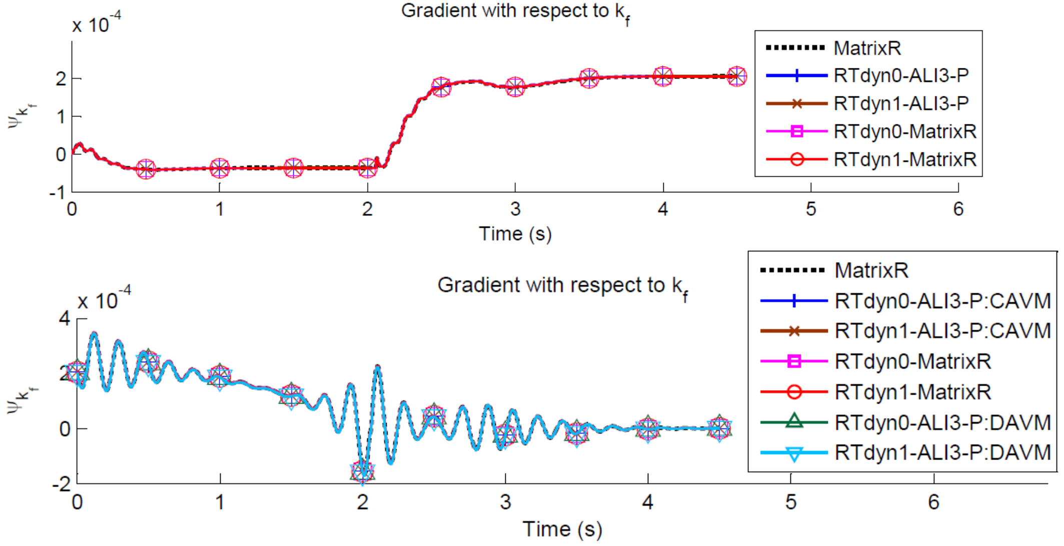


Formulation	CPU time	CPU time natural	Ratio nat/rel
RTdyn0 ALI3-P	2.781	6.703	2.410
RTdyn1 ALI3-P	2.781	6.703	2.410
RTdyn0 MatrixR	5.422	9.781	1.804
RTdyn1 MatrixR	5.422	9.781	1.804

Objective function and parameters:

$$\psi = \int_{t_0}^{t_F} \ddot{r}_{1_z}^2 \, \mathrm{d}t \qquad \boldsymbol{\rho}^{sens} = \left[k_f \quad c_f \quad k_r \quad c_r \quad m_c\right]^{\mathrm{T}}$$

- Sensitivity analysis:



Formulation	CPU time	CPU time natural	Ratio nat/rel
RTdyn0 ALI3-P: DDM	8.188	9.484	1.158
RTdyn0 ALI3-P: CAVM	9.031	10.953	1.213
RTdyn0 ALI3-P: DAVM	8.781	11.047	1.258
RTdyn1 ALI3-P: DDM	8.188	9.484	1.158
RTdyn1 ALI3-P: CAVM	9.031	10.953	1.213
RTdyn1 ALI3-P: DAVM	8.781	11.047	1.258
RTdyn0 MatrixR: DDM	30.156	35.563	1.179
RTdyn0 MatrixR: CAVM	30.350	37.719	1.243
RTdyn1 MatrixR: DDM	30.156	35.563	1.179
RTdyn1 MatrixR: CAVM	30.250	37.719	1.243

Buggy vehicle: design optimization for step descent.

- Minimization problem:

$$\min_{\boldsymbol{\rho}} \quad \psi = \int_{t_0}^{t_F} \ddot{r}_{1_z}^2 \mathrm{d}t$$
$$\text{s.t.} \quad \boldsymbol{\rho} \leq \mathbf{U}$$
$$\boldsymbol{\rho} \geq \mathbf{L}$$

with:

$$\mathbf{L} = \begin{bmatrix} 10 & 10 & 10 & 10 \end{bmatrix}^{\mathrm{T}}$$
$$\mathbf{U} = \begin{bmatrix} 10^5 & 10^5 & 10^5 & 10^5 \end{bmatrix}^{\mathrm{T}}$$

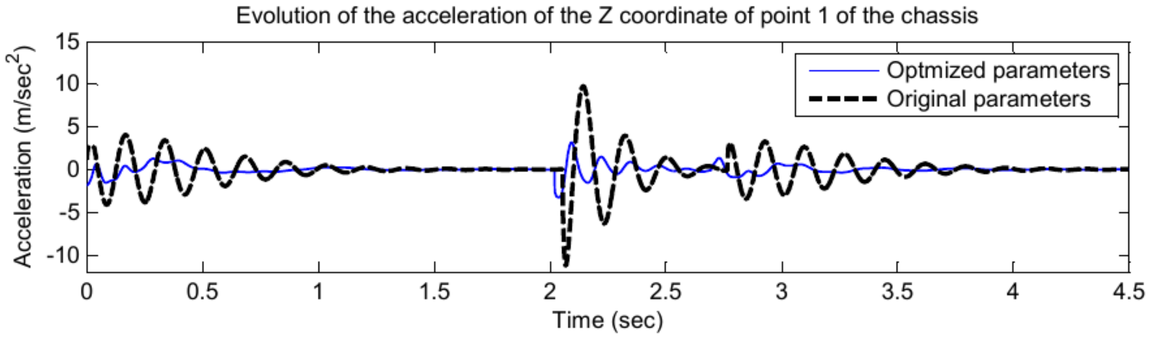
- Optimization parameters:

$$\boldsymbol{\rho}^{opt} = \begin{bmatrix} k_f & c_f & k_r & c_r \end{bmatrix}^{\mathrm{T}}$$

- Optimization algorithms performance (*fmincon*):

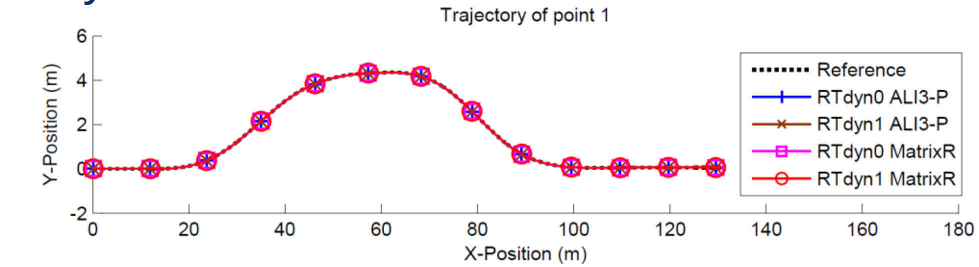
Algorithm	k_f	c_f	k_r	c_r	Ψ	Nº iter
Interior Point	13731.4	245.896	13434.3	59.7709	1.4146	80
TRR	15217.6	7911.09	14514.9	4724.76	13.164	100 (max)
SQP	13886.8	257.488	10962.8	96.9401	1.4601	30
Active Set	13923.9	251.196	10961.4	96.8143	1.4619	19

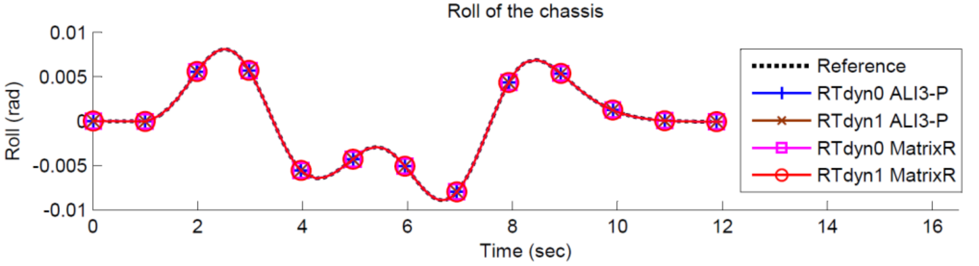
- Optimized response:



Buggy vehicle: DLC maneuver.

Dynamics



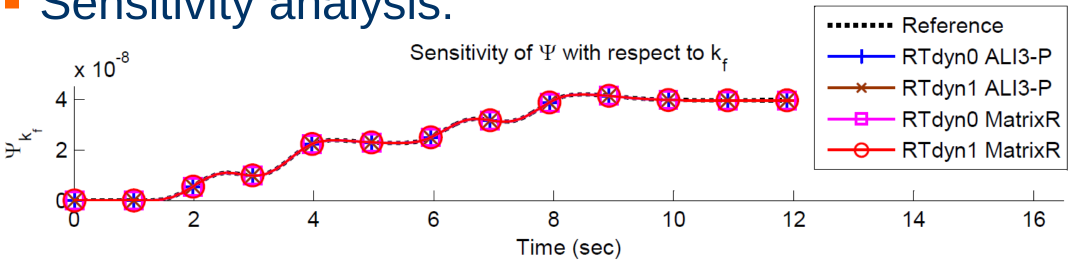


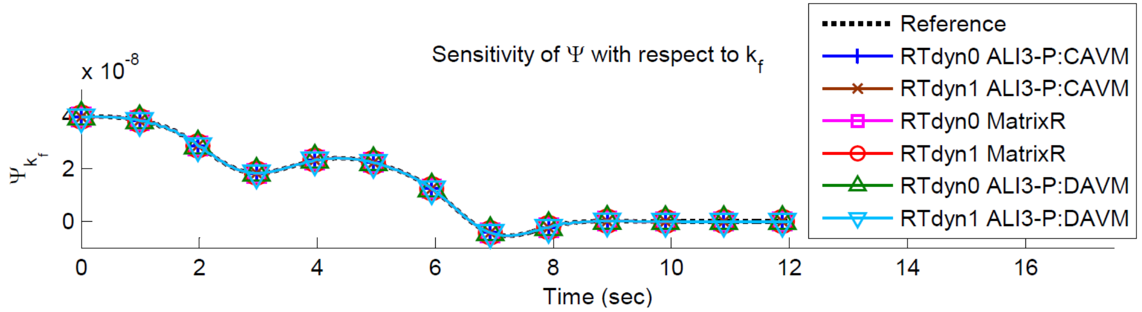
Formulation	CPU time	CPU time natural	Ratio nat/rel
RTdyn0 ALI3-P	1.047	3.750	3.582
RTdyn1 ALI3-P	1.047	3.750	3.582
RTdyn0 MatrixR	2.109	3.859	1.830
RTdyn1 MatrixR	2.109	3.859	1.830

Objective function and parameters:

$$\psi = \int_{t_0}^{t_F} \dot{\phi}^2 \mathrm{d}t \qquad \boldsymbol{\rho}^{sens} = \begin{bmatrix} k_f & c_f & k_r & c_r & m_c \end{bmatrix}^{\mathrm{T}}$$

Sensitivity analysis:





Formulation	CPU time	CPU time natural	Ratio nat/rel
RTdyn0 ALI3-P: DDM	2.578	4.609	1.788
RTdyn0 ALI3-P: CAVM	2.766	-	-
RTdyn0 ALI3-P: DAVM	2.686	4.906	1.827
RTdyn1 ALI3-P: DDM	2.578	4.609	1.788
RTdyn1 ALI3-P: CAVM	2.766	-	-
RTdyn1 ALI3-P: DAVM	2.686	4.906	1.827
RTdyn0 MatrixR: DDM	8.484	11.609	1.368
RTdyn0 MatrixR: CAVM	8.563	11.969	1.404
RTdyn1 MatrixR: DDM	8.484	11.609	1.368
RTdyn1 MatrixR: CAVM	8.563	11.969	1.404

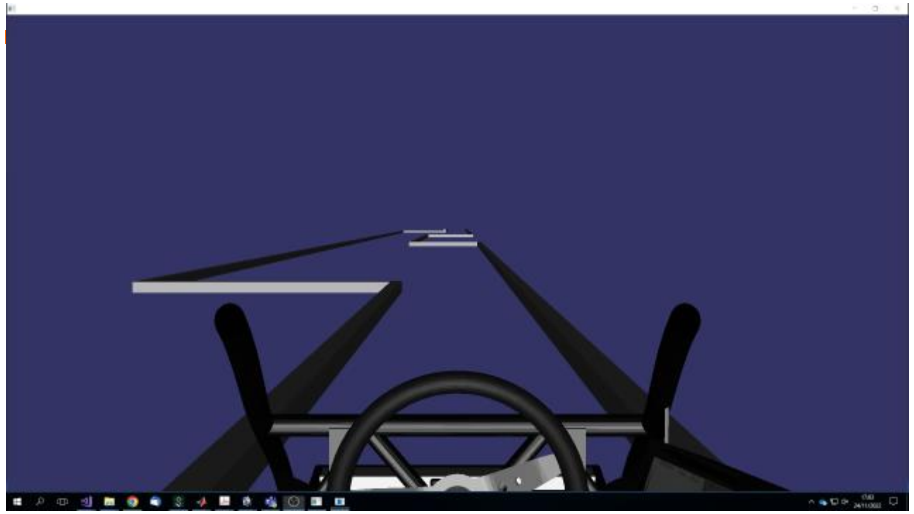
Buggy vehicle: design optimization for DLC.

- Minimization problem:

$$\min_{\boldsymbol{\rho}} \quad \psi = \int_{t_0}^{t_F} \dot{\phi}^2 \mathrm{d}t$$
$$\text{s.t.} \quad \boldsymbol{\rho} \leq \mathbf{U}$$
$$\boldsymbol{\rho} \geq \mathbf{L}$$

- Optimization parameters:

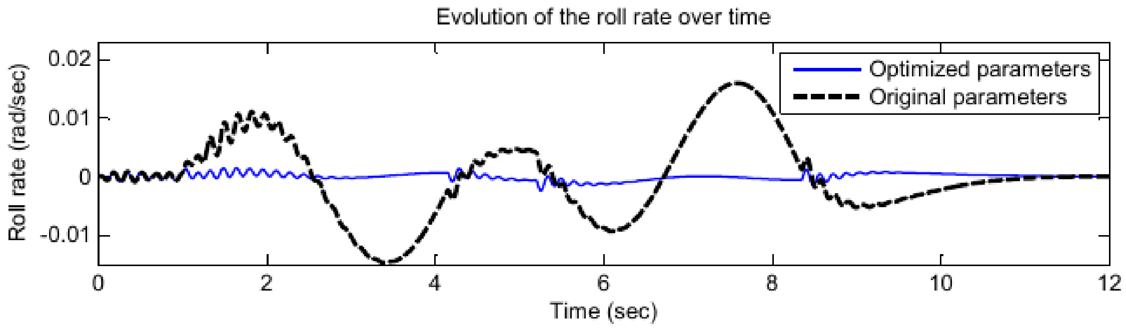
$$\boldsymbol{\rho}^{opt} = \begin{bmatrix} k_f & c_f & k_r & c_r \end{bmatrix}^{\mathrm{T}}$$



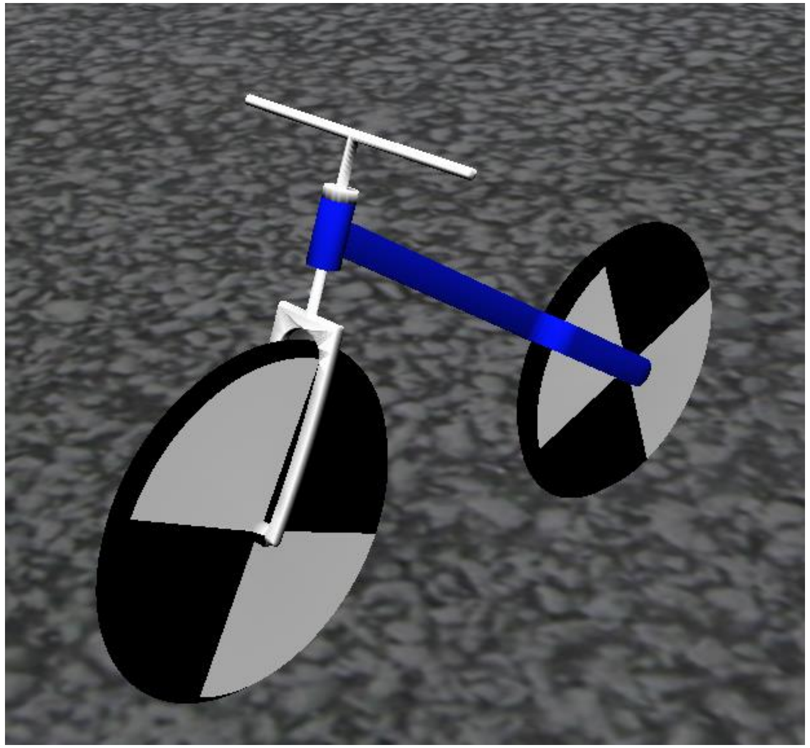
- New minimization problem:

$$\min_{\bar{r}_z^G} \quad \boldsymbol{\psi} = k \int_{t_0}^{t_F} \dot{\phi}^2 \mathrm{d}t$$
$$\text{s.t.} \quad \bar{r}_z^G \leq 1$$
$$\bar{r}_z^G \geq -1$$

- Results:
  - Straightforward optimization.
  - The chassis CoM is lowered to match the center of roll.
  - *Optimized spring-damper coefficients lead to an infeasible solution. Suspensions should be redesigned.



Bicycle: model.

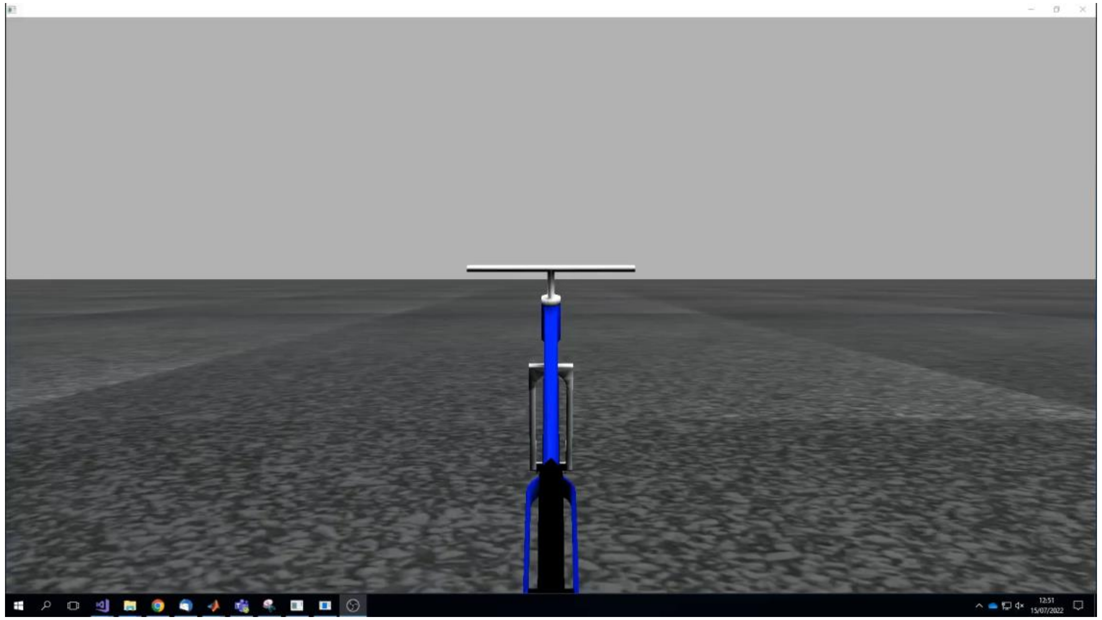


- **Experiment:**
  1. Double turn maneuver during 10 seconds.

- **4 bodies.**
- **9 degrees of freedom.**
- **Forces:**
  - Gravity.
  - Contact-frictional tire forces.
  - Steering torque.
  - Traction torque.
- **Natural coordinates model:**
  - 36 variables.
  - 30 constraints.
  - Modelled with 3 points, 8 vectors and 3 angles.
- **Relative coordinates model:**
  - 10 variables.
  - 1 constraint (Euler parameters).
  - Modelled 1 floating joint and 3 revolute joints.

Bicycle: dynamics and sensitivity.

- Dynamics



Formulation	CPU time	CPU time natural	Ratio nat/rel
RTdyn0 ALI3-P	0.781	3.047	3.901
RTdyn1 ALI3-P	0.781	3.047	3.901
RTdyn0 MatrixR	1.031	3.063	2.971
RTdyn1 MatrixR	1.031	3.063	2.971
RTdyn0 Index-1 FR	0.484	-	-
RTdyn1 Index-1 FR	0.375	-	-

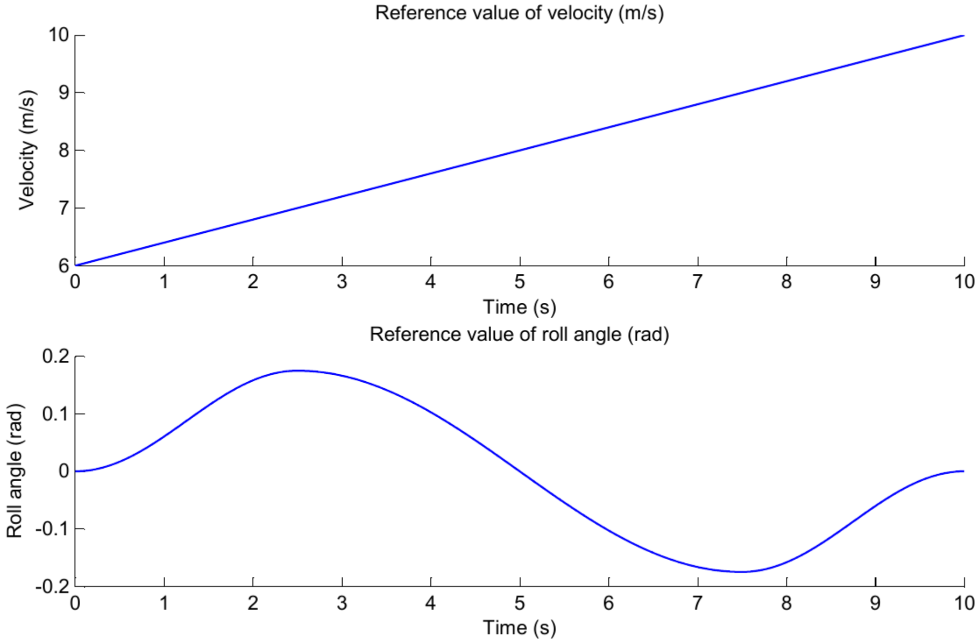
- Sensitivity analysis:

Objective function:

$$\boldsymbol{\psi} = \int_{t_0}^{t_F} \left(\mid \phi - \phi_{ref} \mid + \mid \parallel \dot{\mathbf{r}}_1 \parallel - \dot{r}_{1ref} \mid \right) \mathrm{d}t$$

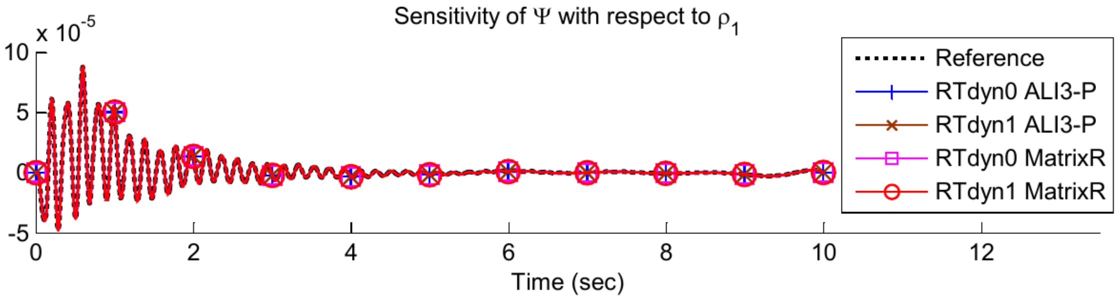
Parameters (spline points):

$$\boldsymbol{\rho}^{sens} = \left[\rho_1 \quad \rho_2 \quad \dots \quad \rho_p \right]^{\mathrm{T}}$$



Bicycle: sensitivity analysis.

- Sensitivity analysis with 32 parameters:



Formulation	CPU time	CPU time naturals	Ratio nat/rel
RTdyn0 ALI3-P: DDM	1.967	10.063	5.116
RTdyn0 ALI3-P: CAVM	1.750	5.578	3.187
RTdyn0 ALI3-P: DAVM	1.718	5.469	3.183
RTdyn1 ALI3-P: DDM	2.078	10.063	4.843
RTdyn1 ALI3-P: CAVM	1.891	5.578	2.950
RTdyn1 ALI3-P: DAVM	1.844	5.469	2.966
RTdyn0 MatrixR: DDM	6.219	8.484	1.364
RTdyn0 MatrixR: CAVM	6.141	9.203	1.499
RTdyn1 MatrixR: DDM	6.313	8.484	1.344
RTdyn1 MatrixR: CAVM	6.219	9.203	1.480

- Effect of an increment of the number of parameters:

	Number of parameters				
Formulation	32	100	200	500	1000
RTdyn0 ALI3-P: DDM	1.906	2.406	3.219	5.359	9.203
RTdyn0 ALI3-P: CAVM	1.734	1.938	2.047	2.328	2.953
RTdyn0 ALI3-P: DAVM	1.703	1.766	1.906	2.266	2.922
RTdyn1 ALI3-P: DDM	2.063	2.578	3.375	5.813	10.375
RTdyn1 ALI3-P: CAVM	1.813	1.969	2.219	2.781	3.906
RTdyn1 ALI3-P: DAVM	1.781	1.984	2.189	2.813	3.875
RTdyn0 MatrixR: DDM	5.969	7.422	9.891	17.672	30.391
RTdyn0 MatrixR: CAVM	5.890	7.172	9.641	16.875	28.953
RTdyn1 MatrixR: DDM	6.063	7.563	10.172	18.125	31.359
RTdyn1 MatrixR: CAVM	6.016	7.375	9.781	17.438	29.500

Outline

1) Introduction and motivation
2) Dynamics of open-loop systems
3) Dynamics of closed-loop systems
4) Sensitivity analysis of unconstrained open-loop systems
5) Sensitivity analysis of closed-loop systems
6) MBSLIM implementation
7) Numerical experiments
8) Conclusions

Conclusions and future work.

- Conclusions about the methods:
  - A **more general and systematic description** of the already existing topological semi-recursive methods has been provided.
  - **Semi and fully-recursive formulations** for unconstrained open-loop systems have been covered.
  - **Semi-recursive methods** have been combined with two constraint enforcement techniques (**Matrix R** and **ALI3-P**).
  - The **forward sensitivity analysis** of unconstrained dynamic formulations have been addressed.
  - **Forward and adjoint sensitivity formulations** have been developed for constrained dynamic problems.
  - **Derivatives** of recursive kinematic relations, accumulations and assembly procedures have been attained by means of **analytical differentiation.**
- Conclusions about the implementation:
  - **Kinematics, dynamics and sensitivity analysis** of joint coordinate models are now supported by MBSLIM.
  - The new **implementation is general**, this is, it supports almost any type of multibody system.
- Conclusions about the numerical experiments:
  - Recursive formulations display a **high accuracy.**
  - Recursive dynamics and sensitivity analysis could be **more computationally efficient** than global methods.
  - The analytical sensitivity formulations have been successfully applied to **optimization problems**.

Conclusions and future work.

- Future work:
  - **Flexible multibody formulations** (ongoing).
  - **Fully-recursive** formulations and sensitivities for closed-loop systems.
  - Deeper study of **gradient-based optimization methods.**
  - **Increase efficiency** of recursive formulations.

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- Journal papers:
  - D. Dopico Dopico, A. López Varela, A. Luaces Fernández. Augmented Lagrangian index-3 semi-recursive formulations with projections. *Multibody System Dynamics*, 2020. doi: 10.1007/s11044-020-09771-9.
- Submitted journal papers:
  - A. López Varela, C. Sandu, A. Sandu, D. Dopico Dopico. Discrete Adjoint Variable Method for the Sensitivity Analysis of ALI3-P Formulations. *Multibody System Dynamics*.
- Journal papers in development:
  - A. López Varela, D. Dopico Dopico, A. Luaces Fernández. Direct sensitivity analysis of semi-recursive ALI3-P formulations.

Conclusions and future work.

Conference communications:

- A. López Varela, D. Dopico Dopico, A. Luaces Fernández. Direct sensitivity analysis of multibody systems modeled with relative coordinates using an augmented Lagrangian formulation with projections. In *Proceeding of the ASME 2020 International Design Engineering Technical Conferences and Computers and Information in Engineering Conference - 16th International Conference on Multibody Systems, Nonlinear Dynamics and Control*. Saint Louis, USA (virtual conference), 2020.
- A. López Varela, A. Luaces Fernández, D. Dopico Dopico. Adjoint sensitivity analysis of multibody systems modeled with joint coordinates using an augmented Lagrangian formulation with projections. In *Proceeding of the ASME 2021 International Design Engineering Technical Conferences and Computers and Information in Engineering Conference - 17th International Conference on Multibody Systems, Nonlinear Dynamics and Control*. Saint Louis, USA (virtual conference), 2021.
- A. López Varela, A. Luaces Fernández, D. Dopico Dopico. Discrete adjoint approach for the sensitivity analysis of an augmented Lagrangian index-3 formulation with projections. In *10th ECCOMAS Thematic Conference on Multibody Dynamics*. Budapest, Hungary (virtual conference), 2021.
- D. Dopico Dopico, A. López Varela, A. Luaces Fernández. Optimization of a three wheeled tilting vehicle. In *10th ECCOMAS Thematic Conference on Multibody Dynamics*. Budapest, Hungary (virtual conference), 2021.
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- A. Luaces Fernández, A. López Varela, A. Verulkar, C. Sandu, A. Sandu, D. Dopico Dopico. Optimal design and control of the steering of a tilting bicycle. In *6th Joint international conference on multibody system dynamics (IMSD). New Delhi, India, 2022.*

Sensitivity analysis and optimization of the dynamics of multibody systems using analytical gradient based methods

Thank you for your attention!

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